Your Metaphor, or My Metonymy?

A Response to Elizabeth Mowat & Brent Davis

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Elizabeth Mowat and Brent Davis have provided a stimulating proposal concerning the complexity of understanding mathematics. Their approach includes ways to probe and delve into relationships between mathematics and both its pedagogy and its didactics,[1] by means of the structure of a dynamic network. My main comment is that they have only just begun turning over the surface leaves, and that a great deal more in-depth probing is likely to yield further treasures. Because metaphor is so central to the proposed network structure, in what follows I shall irritatingly stress multiple interpretations of the copulative verb *is*.

The authors propose that mathematics can be considered as (in metaphoric terms, *is*; in terms of a simile, *is like*; in model terms, *can be modelled by*, or *can usefully be thought of in terms of*) a dynamically developing network. What then are (are usefully considered to be or to be like) the nodes and connections? The authors opt for *conceptual domains* as nodes, and *metaphors* as connections (Mowat & Davis, p. 10-12).

[1] I choose to distinguish between pedagogic strategies that can be used in many or most mathematical topics, and didactic tactics which are specifically to do with a single concept, technique, theorem, proof, etc.

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Nodes

Exactly what constitutes a conceptual domain is quite hard to pin down in these early stages of delineating the network. Examples of nodes proposed include arithmetic, the container-image-schema and functions. Now these three examples encompass a broad range of "things". Arithmetic is made up of multiple actions performed on collections (containers) of objects and by means of which other collections are constructed. Numbers arise as indicators or partial attributes of collections, and then actions are transferred (carried across metaphorically) from the collections to the indicators. The result is that arithmetic is a product of metaphorical transference, commonly called abstraction. Functions emerge as awareness of the structure of actions (they act upon something and they produce a result), which itself is a fundamental psychological observation with Pythagorean and Samkhya origins.

Conceptual domains are also taken to include *image schemas* such as the four grounded metaphors of Lakoff & Núñez (2000), namely object collection, object construction, measure and motion on a path. These four provide the basis, so the claim goes, for all of mathematics. Everything else is formed, so they claim, from conceptual blending (à lá Fauconnier & Turner 2002) and further metaphors between abstractions.

It is easy to locate container metaphors whenever sets, families or collections are mentioned, but this is too simplistic to make a convincing case that network theory has something to add to the rich web of mathematics education constructs. The same is true concerning the use of functions, which pervade modern mathematics. It is not clear how the simple observation that mathematics in its modern expression subsumes and builds from sets and functions (though indeed there are set-less formulations) really provides insight into understanding of advanced concepts in the higher reaches. The Lakoff-Núñez claim that all of mathematics can be tracked back to body-based metaphors is not helpful if it rests solely on the simplistic observation of the role of sets and functions.

It is not, however, always easy to find the bodily basis for more sophisticated mathematical constructs. For example shears are perhaps best exemplified by easing a deck of cards or a sheaf of papers so that they can more readily be separated by the thumb. This is the basis behind Cavalieri's approach to integration, but it is really only second-hand bodily experience: it is the cards that shear, not our bodies. Shears emerge from mathematical analysis of possible affine transformations, once the invariants that constitute an affine transformation are identified. Even farther from bodily experience are constructs abstracted from abstractions from abstractions of bodily experiences. Just as the definition of a continuous function opened up a world of unexpected objects such as the van Koch snowflake (no-where differentiable and somewhat beyond human comprehension), the space-filling Hilbert curve (whose basis in bodily experience seems to be somewhat remote), the notion of dimensions being positive *reals* not whole numbers, (arising from Hausdorff measures and fractals and very difficult to relate directly to bodily experience) and the manipulation of transfinite cardinals, so in many conceptual domains in mathematics extensions of meaning are made by weakening constraints and trusting logic applied to requiring consistency with current axioms or properties. The fact that these extensions and restrictions are then cast in metaphoric

language ("a fraction is a number") might actually divert attention from the pervasive theme of how extending and restricting meaning is accomplished. It certainly involves metaphor (carrying meaning across) but it has other psychological and logical components as well. Even such a basic abstraction as exponentiation (of a positive real number) arises from a parallel movement to the arising of number: indicators (powers) are isolated, abstracted, and then released from the restrictions of being whole numbers so as to extend to rationals and then to reals.

Connections & Links

On first reading, there seems to be some confusion as to whether the container metaphor is a node or a link, but careful reading reveals that the links in the network are to be taken as the assertion "is a", so the links pointed to "are" the metaphors as bearers of meaning and logical relations. The notion of conceptual blending suggests that the type of connections being considered properly belong in hypergraphs where three or more nodes are combined in a single "face" or simplex.

Basing links solely on metaphors requires that considerable care be taken to select the appropriate relationships of the metaphor-source to carry across to the metaphor-target. This is the role of analogy, originally a mathematical term for what we now call proportion, in which correspondence between details and structural relationships is elaborated in the source and target domains. So behind each metaphoric link is the detail of an analogy. But there might be other ways in which people make connections as well.

Reflecting on the way concepts, theorems and practices come to mind, I find that there are themes which pervade mathematics, such as *invariance in the midst of change*, *doing & undoing* (inverse actions), *freedom & constraint*, and *extending & restricting attention* which also serve to hold together otherwise apparently disparate elements. To return to the last of these, extending & restricting meaning is invoked when attention is focused, for example on the topology induced on a subset of a topological space, or the subgroup generated by a subset of group elements or the extension of number to include fractions. It arises especially when attention is directed to a subfield of the *reals* which "*is*" also an extension by some irreducible polynomial, or when an extra constraint is added to a set of axioms or imposed on a definition to yield a subclass. As already acknowledged, there is metaphoric action present here, but perhaps there is something more as well. Mathematical themes could perhaps be cast as metaphors or instantiations of image schemas, but the advantage gained by theorising encompassing multiplicity under succinct labels may obscure the opportunity to distinguish types of connections.

Many authors, such as Pimm (1995), Lakoff & Johnson (1980), Sfard (1994) and Presmeg (2005) among others, in addition to Lakoff & Núñez (2000), not to say Jakobson (1951) have pointed out that the grammatical structure of metaphor underpins how we express connections, including, as in the present case, the influence of mathematical themes. But metaphor is not everything, or rather, everything is not metaphor. As Jakobson pointed out (and Lakoff & Johnson acknowledged), there are also idiosyncratic playful associations (contiguities) between ideas based on various forms of metonymy,

such as synechdoche (part for whole), metalepsis (reference by remote association) and homonymy, not to say idiosyncratic and largely emotionally triggered connections. Metonymies "trip along the surface", depending on homonyms, puns, and emotional connections, whereas metaphors are more structural and hence "deep" (note the inevitable use of metaphors for talking about metaphor and metonymy). Metonymies are powerful connectors for human beings, experienced as associations that "come to mind unbidden". These too might eventually be incorporated in an even more complex network of nodes and connections.

In following Lakoff & Núñez closely, it is difficult not to keep returning to the basic metaphors of containers, and object-construction as core connectors. For network theory and complexity to be a powerful addition we need to be clearer about the interplay between metaphors, analogies, metonymies and other forms of connections. Much of the influence that mathematics-as-network can offer is in sensitising teachers to how human beings "have something come to mind", that is, how connections are made in the moment. An important form is through frozen or deeply embedded metaphors that might easily be blocking learner sense-making through the learner being.

Layers

Since conceptual domains are (can themselves by considered to be) networks of further nodes and connections, the network being adumbrated is (can be seen as being) formed of multiple layers: subjective mathematical understanding of the individual, of a classroom collective, as portrayed in a curriculum, and as formal Popperian-third-world mathematics (Davis & Simmt, 2006), among perhaps others. It is straightforward to see how subjective understanding is dynamic and evolving, being in flux through growth and through pruning. It is not too difficult to see that a class that meets regularly establishes an ethos, ways of working, and a collective grasp comprised not simply of the ways in which individuals contribute to a whole greater than their individual understanding, but in their apparently shared tacit understandings, attested to by their proleptic interactions, and the collective "sense of the whole group" which the teacher may partially read. It seems evident that this collective understanding is also dynamic and evolving, comprised as it is of both individual's changing conceptions and connections, and the influence this has on the collective and vice versa. Mathematics as portrayed by curriculum statements also changes over time, as can be seen from historical documentation. Formal mathematics itself grows and develops in response to insights and fresh problems, as can be seen from the shelves of any mathematics department library, or by tracing the rise and fall of topics such as the umbral calculus. A striking feature of these layers is that the closer to the individual, the greater the dynamic; the closer to the formal, the greater the inertia, as one would expect (Mowat & Davis, p. 7).

One of the difficulties in thinking in this multi-level/multi-nodalway is that it is hard to remain on one level. In a sense, each identified and hence ontologised entity in mathematics could well be a node at any of the levels. Thus the conceptual fields

(Vergnaud, 1997) of additive and multiplicative reasoning contribute to arithmetic at the individual, collective, curriculum and formal level. Specifying an appropriate grain size for different levels is a more complex task than it first seems, and certainly deserves further attention.

Subjective Understanding Network

The network under consideration is the layer of personal subjective understanding of mathematics. At one point in their paper the adjective *neural* appears, suggesting that the network under consideration may have some ontological relationship with the neural network of the brain. This might be the case, but evidence is not likely to be available for a long time yet. Indeed, it seems to me that any "complete" metaphor-model for interconnections as enacted in human brains must take on a wider class of nodes and types of connections than the authors presently wish to contemplate.

Flexibility & Multiplicity

The dynamic and co-emergent nature of mathematics (despite public conception otherwise), and especially of subjective and collective mathematical understanding and appreciation, requires flexibility, a form of metaphoric floating redolent of Mohammed Ali's "float like a butterfly, sting like a bee". Unfortunately we also know that pre-adolescents, especially boys, often have a difficult time letting go of their addiction to absolutes, certainties and what they consider to be "direct, tell-it-like-it-is" speech. When they discover that things are not exactly as they have been told, that there are shades of gray and that assertions, especially in literature can have multiple meanings, they sometimes dig their heels in and focus attention on sport and mathematical procedures, believing that these are domains of certainty. Rarely is their attention drawn to the deeply metaphoric structure of their current idioms. Consequently, the inescapable metaphor-analogy-simile nature of human communication may not be readily absorbed by all learners at first. There are open questions about how to judge the degree of explicitness of multiplicity of meaning in mathematics at various ages and stages of intellectual growth.

Aspects Particularly Worthy of Further Probing

What are the mechanisms that "bring an action to mind"? Minsky (1975) proposed the notion of default parameters in "frames", and the authors refer to Lamb (1999) for something similar. Is there a difference between metaphoric resonances (Richard Skemp used the analogy of humming in front of a grand piano) and metonymic triggers? Are there other distinct ways, with or without grammatical analogues? For example, in his groundbreaking book Felix Klein (1932) showed how functions, trigonometry, groups and geometry could be unified using the notion of the Riemann sphere with complex numbers. Here the connections might, or might not, be functioning exactly the way

Lakoff & Núñez claim concerning metaphors. One person's metaphor can sometimes be another person's metonymy!

Functions, particularly morphisms that carry structure, served twentieth century mathematics well. Might analogy be a significant contribution to an extension of metaphor as what makes connections valuable?

What usefully constitutes a node at different grain sizes or levels of the network? Furthermore, does it make sense to think of hypergraphs rather than graphs when thinking of connections between nodes, and what might this say about robustness and the undoubtedly complex process of learning mathematics?

As networks are elaborated, might it be the case that the notion of being scale-free might turn out to be a little more complicated? It might be that there are subtle differences in the interconnectivities of individual understandings, and those of formal mathematics and working mathematicians. These issues certainly deserve further elaboration and probing.

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