Canadian Studies in Population, Vol. 22(1), 1995, pp. 67-78.

Boundary Condition Sets Limits on Brass's Model Life Table Parameters

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Abstract

The Brass mortality model is based on a strong linear relationship between the logits of the survivorship probabilities of any two life tables. This strong empirical relationship justifies the construction of model life tables from any life table taken as a standard, by manipulating the values of the parameters of the linear equation. The purpose here is to show that a boundary condition exists for this model which has so far gone unnoticed. The simple exercise of fitting and testing the goodness of fit of the model does not reveal this boundary condition. That can only be found through an investigation of other forms of relationships between life tables which must also be true if the linear model holds. One such investigation leads to the restriction that the slope of the Brass model must be equal to one. This restriction not only reduces the number of parameters of the linear equation from two to one but also allows the equation an alternative and simpler expression. It can be shown that the reciprocals of the survivorship functions are linearly related with the restriction that the slope and the intercept coefficients must add up to one. Empirical tests of this revised model seem quite encouraging, and suggest scope for further improvement.

Résumé

Le modèle de mortalité Brass est basé sur une forte relation linéaire entre les logits des probababilités de survie de deux tables de vie quelconques. Cette forte relation empirique justifie la construction de tables de vie modèles à partir de toute table de vie prise à titre de norme, en manipulant les valeurs des paramètres de l'équation linéaire. L'objectif est ici de démontrer l'existence d'une condition-limite pour ce modèle, laquelle n'a pas été notée jusqu'ici. Le simple exercice consistant à ajuster et à tester le modèle ne révèle pas cette condition. Elle ne peut être trouvée qu'au terme d'une investigation portant sur d'autres formes de relations entre les tables de vie, qui doivent également être vraies si le modèle linéaire tient. Une investigation de ce type aboutit à la restriction dictant que la pente du modèle Brass doit être égale à un. Cette restriction ne limite pas seulement le nombre de paramètres de l'équation linéaire de deux à un, elle donne également lieu à une autre expression plus simple. On peut démontrer que les réciproques des fonctions de survie sont liées de façon linéaire, à condition toutefois que la pente et les coefficients de la coordonnée à l'origine donnent un. Les tests empiriques de ce modèle révisé semblent prometteurs et pourraient donner lieu à d'autres améliorations.

Keywords: life table survivorship function, logits, nonlinear, trial solution, goodness of fit

The Problem

In an empirical demonstration, Brass (1975) has shown that the logits of any two life table survivorship functions are highly correlated and accordingly, the relationship between the two can be expressed in terms of a linear equation as

$$\ell n\left(\frac{1-\ell_0(x)}{\ell_0(x)}\right) = a + b\ell n\left(\frac{1-\ell_s(x)}{\ell_s(x)}\right) + \epsilon \tag{1}$$

In (1) $\ell_0(x)$ and $\ell_s(x)$ are the observed or the given probabilities of survival from birth to age x in the two life tables and $\in$ stands for the error component. Therefore, from any given or standard life table with known $\ell_s(x)$ values, other life tables can be generated for appropriate combinations of the values of the parameters a and b. The parameters have been found to vary in a systematic manner with levels of mortality such that in most cases a decreases and b increases with increase in life expectancy (Keyfitz, 1991).

The purpose of this note is to demonstrate the existence of a boundary condition in (1) which has hitherto gone unnoticed. What follows next is the derivation of this condition and the restriction it imposes on the parameters a and b.

The Boundary Condition

To that end, we begin by dropping the error term $\in$, replacing ℓ_0(x) by its expected value ℓ(x) and differentiating both sides of (1) with respect to x. The operation produces

$$\frac{\ell(x)\mu(x)}{1-\ell(x)} + \mu(x) = b\left[\frac{\ell_s(x)\mu_s(x)}{1-\ell_s(x)} + \mu_s(x)\right] \tag{2}$$

where

$$\mu(x) = -\frac{1}{\ell(x)}\frac{d\ell(x)}{dx} = -\frac{d\ell n\ell(x)}{dx} \tag{3}$$

is the force of mortality at age x corresponding to the first life table with $\ell(x)$ as the survivorship function at age x. A similar expression can be written with $\ell_s(x)$ for the force of mortality $\mu_s(x)$ of the standard table. Simplifying (2), we get

$$\frac{\mu(x)}{1-\ell(x)} = b\frac{\mu_s(x)}{1-\ell_s(x)} \tag{4}$$

or

$$\frac{\mu(x)}{\mu_s(x)} = b\frac{1-\ell(x)}{1-\ell_s(x)} \tag{5}$$

Since (5) holds for all x, it must also be true for $x = 0$ at which the right hand side of (5) assumes the indeterminate form of 0/0 since $\ell(0) = \ell_s(0) = 1$. It can, however, be determined from its limiting value as $x \to 0$ by applying the well known L'Hospital's rule. Thus, we first write the limiting value of (5) as

$$\lim_{x\to 0} \frac{\mu(x)}{\mu_s(x)} = b \lim_{x\to 0} \frac{1-\ell(x)}{1-\ell_s(x)} \tag{6}$$

Next, we differentiate the numerator and the denominator of the righthand side of (6) with respect to x to get the limiting value as

$$\frac{\mu(0)}{\mu_s(0)} = b\frac{\mu(0)}{\mu_s(0)} \tag{7}$$

from which we get

$$b = 1 \tag{8}$$

The boundary condition set by (5) at $x = 0$ therefore reduces the number of parameters of the Brass model from two to only one.

The Reduced Model

Note that with $b = 1$, (5) can be written as

$$\frac{\mu(x)}{\mu_s(x)} = \frac{1-\ell(x)}{1-\ell_s(x)} \tag{9}$$

An alternative expression for (1) can next be derived by first writing

$$a = \ell nk \tag{10}$$

In that case, given $b=1$, (1) can be written as

$$\frac{1-\ell(x)}{\ell(x)} = \frac{k(1-\ell_s(x))}{\ell_s(x)} \tag{11}$$

or

$$\frac{1}{\ell(x)} = (1-k) + \frac{k}{\ell_s(x)} \tag{12}$$

Equation (12) describes a linear relationship between the reciprocals of the survivorship functions of two life tables which of course meets the boundary condition at $x = 0$ as it should. It may also be noted that the slope and the intercept coefficients of the straight line add up to unity. Note from (10) that k is always nonnegative although it may be greater or less than unity, depending on whether the mortality rates of the standard life table are less or greater than those of the other. This may be seen by rearranging (12) as

$$\frac{1}{\ell_s(x)} = (1-\frac{1}{k}) + \frac{1}{k}\cdot\frac{1}{\ell(x)} \tag{13}$$

which expresses the reciprocal of $\ell_s(x)$ as a linear function of the reciprocal of $\ell(x)$. Here the slope coefficient turns out to be the reciprocal of k. Thus, the model does not allow for any crossover between the survivorship functions of any two life tables, which has become a controversial issue in recent times (Nam, 1993). This may also be seen by writing (12) as

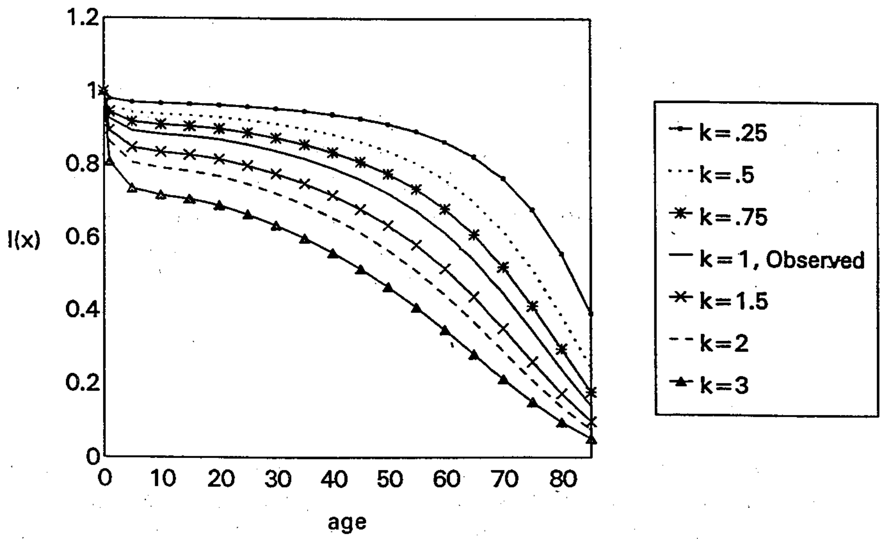
$$\frac{\ell_s(x)}{\ell(x)} = (1-k)\ell_s(x)+k \tag{14}$$

from which it can be seen that for all x, $\ell(x) > \ell_s(x)$ when $0 < k < 1$ and $\ell(x) < \ell_s(x)$ when $k > 1$. Incidentally, a rectangular distribution for $\ell(x)$ is obtained when $k=0$. Since the equation that results after differentiation of both sides of (12) with respect to k, is

$$\frac{d}{dk}\left(\frac{1}{\ell(x)}\right) = -1+\frac{1}{\ell_s(x)} \geq 0 \tag{15}$$

it follows that $\ell(x)$ is a decreasing function of k for any given $\ell_s(x)$. This may be seen in Figure 1 where the model $\ell(x)$ functions have been derived for several values of k.

Figure 1. Observed and Expected l(x) Values for Varying Levels of k. Mexico (1970) is the Standard Table.



Note also, that the model does not allow for any crossover for the mortality rates either. This follows from (9) according to which for all x, $\mu(x) < \mu_s(x)$, when $\ell(x) > \ell_s(x)$ and vice versa.

An investigation of the nature of the model survivorship functions at the other end of the life span seems to be in order at this point. For that we rewrite (14) as

$$\ell(x) = \frac{\ell_s(x)}{k + (1-k)\ell_s(x)} \tag{16}$$

Observe that both $\ell(x)$ and $\ell_s(x)$ must vanish together meaning that the life span remains the same at all levels of mortality which should not be viewed as an unrealistic restriction on the model. Incidentially, the restriciton may be seen to hold on the original Brass model as well (see eq. 1).

Estimates of k

In order to estimate the parameter k we first rewrite (12) as

$$\frac{1}{\ell(x)} - 1 = k\left(\frac{1}{\ell_s(x)} - 1\right) \tag{17}$$

and then find the line of best fit of $1/\ell_0(x)-1$ on $1/\ell_s(x)-1$ with zero intercept. It is quite possible that the estimate of k thus obtained may not simultaneously minimize the sum of squares of differences between the given and the estimated values of $\ell_0(x)$. Consequently, we have taken the next step towards that goal by using either of those estimates of k as the initial or trial solution for the standard computational program for nonlinear regression. At this point, it is felt that since the variance of $\ell_0(x)$

$$V(\ell_0(x)) = \ell_0(x)(1-\ell_0(x)) \tag{18}$$

is a function of x, the minimization of the sum of squares may preferably be carried on with weights proportional to the reciprocal of the respective variances. The operational procedure for the estimation of k therefore, requires the minimization of

$$E = \sum (\ell_0(x) - \ell(x))^2/(\ell_0(x)(1-\ell_0(x))) \tag{19}$$

In practice however, the best solution of k may be obtained from (19) without either of the trial solutions. An arbitrary initial value of 0.7 for example for

k is enough for the SPSS nonlinear regression program to find the best solution.

Experiment with Models

Life tables of three countries covering the wide range of life expectancy were selected and were subjected to graduation by Brass's two parameter, as well as the present single parameter, model. The experiment was repeated twice by selecting two different standard life tables covering the range of life expectancies. See Table 1 for values of the parameters and information about the standards used.

Table 1. Estimates of the Parameters of the One-Standard and Two-Standard Models.

	One-Standard Model		*Two-Standard Model*	
Standard Country (Males)	*Guatemala (1972-73, 53.8)*	*England & Wales (1982-84, 71.5)*	*Both Guatemala and England and Wales*	
	k	k	c	d
Botswana (1980-81, 52.7)	1.06281	6.70403	.83795	1.12307
Bahrain (1976-81, 63.3)	.52507	2.39343	.32388	.88029
Japan (1984, 74.4)	.16551	.67599	.02397	.58906

Note: Figures inside parenthesis are year and life expectancy in that order.
Source: United Nations. 1985 Demographic Yearbook.

Not unexpectedly, the quality of fit of the model to the actual life tables seems to depend on the closeness of the given life table with the standard life table. This may be seen in Figure 2 for the three life tables for males. The female comparison is quite similar and is not shown here.

Figure 2. Observed and Expected l(x) Values from the Brass and One-Standard Models for Selected Male Life Tables.

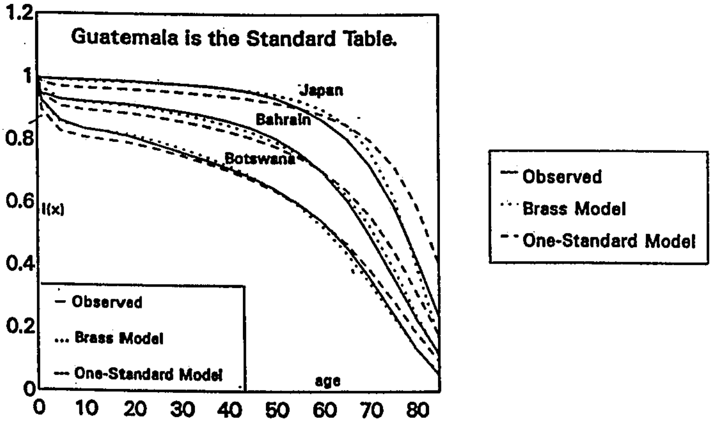
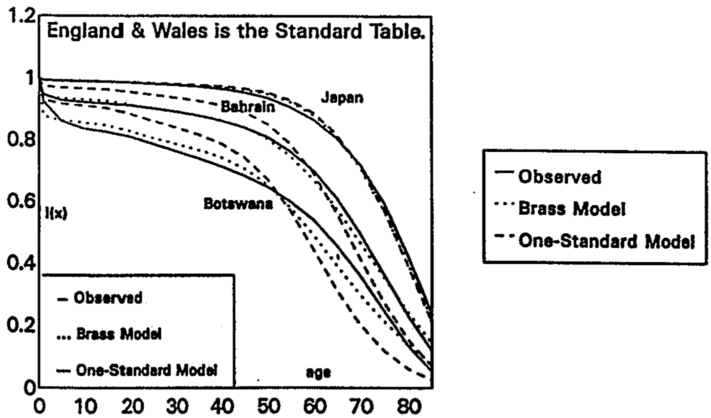


Figure 2 (continued). Observed and Expected l(x) Values from the Brass and One-Standard Models for Selected Male Life Tables.



This is true for both models, although Brass's two-parameter model has a slight edge over the one parameter model in spite of its failure to meet the boundary condition. Accordingly, we chose to enlarge the one parameter to a two parameter model by entering two instead of one standard life table without altering the functional form of the equation. This can be accomplished by defining a two-standard model as a linear compound of two single standard models given by

$$\frac{1}{\ell(x)} - 1 = c\left(\frac{1}{\ell_s(x)} - 1\right) + d\left(\frac{1}{\ell_t(x)} - 1\right) \tag{20}$$

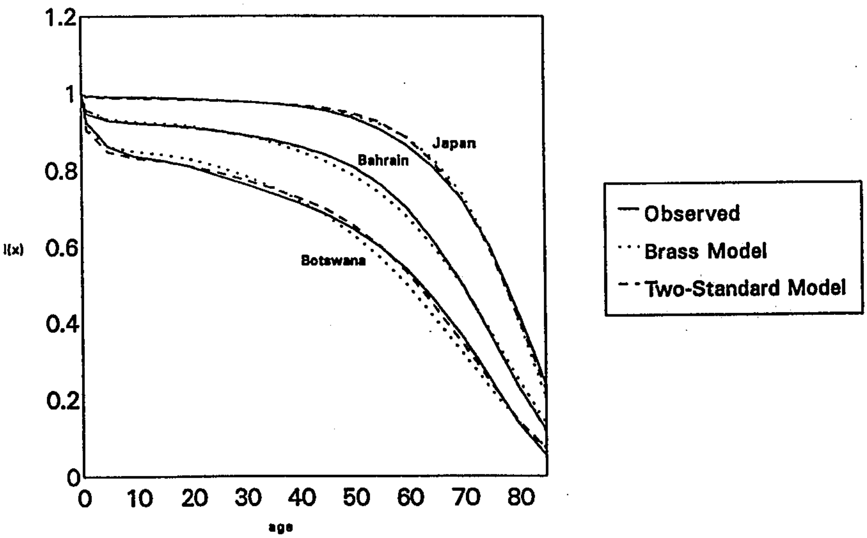
where $\ell_s(x)$ and $\ell_t(x)$ stand for the survivorship functions of two standard tables. The justification of such a procedure stems from the fact that the function $1/\ell(x)$ of any two life tables are highly correlated. Consequently, the multiple correlations between $1/\ell(x)$ of any life table and a linear compound of the same of any other k life tables must be a nondecreasing function of k. As will be shown later, $k=2$ is sufficient for the present exercise since it brings that multiple correlation sufficiently close to unity.

As before, we proceed with the estimation of the parameters c and d by the method of least squares for linear regression with zero intercept. The estimates of the parameters c and d are then used as trial solutions to minimize E as shown in (19).

As noted earlier, the one-standard or the single parameter model produces a good fit when the levels of mortality of both the standard and the given life tables do not differ greatly from one another. Common sense therefore, dictates that the best result from the two-parameter model can be obtained when the two-standard tables are selected from the two ends of the life expectancy continuum. Accordingly, Guatemala, with a life expectancy of 54 years, and England and Wales, with a life expectancy of 73 years were selected for testing the two-standard model. The quality of fit may be seen in Figure 3 where for purposes of comparison, the logit model of Brass has also been shown for which Mexico, with a life expectancy of 63 years, has been chosen as standard. The choice of Mexico is guided by the fact that its life expectancy is close to the average of the same of the countries selected for the two-standard model.

Clearly, the two-standard model seems to produce a better fit than the Brass logit model with both having two parameters.

Figure 3. Observed and Expected l(x) Values from the Brass (Standard is Mexico) and Two-Standard (Guatemala, and England and Wales) Models for Selected Male Life Tables.



In order to judge the quality of a model, one can come up with several objective measures of goodness of fit. One such measure can be generated from the index that Heligman and Pollard (1980) used to minimize, in order to estimate the parameters of their model curve for probabilities of dying. An equivalent version of that index in our example will be

$$s^2 = \sum \left(\frac{\ell_0(x)}{\ell(x)} - 1\right)^2 \tag{21}$$

As such, $s2$ provides a measure of departure of the observed from the expected values of survivorship function in relative terms, although its magnitude depends on the number of data points n used for its derivation. Also, the measure being comparable to the variance is not linear. Accordingly, we have chosen to define a measure such as root mean square deviation given by

$$e = s\sqrt{n} \tag{22}$$

which is linear as well as independent of the number of data points.

In general, the quality of a model can usually be studied by constructing an index that measures the equality of the observed and the expected values. Certainly, *e* accomplishes that objective to a certain extent. It assumes the smallest value of zero when the fit is perfect but unfortunately its upper boundary is indeterminate. Accordingly, we have chosen to provide another measure which has earlier been used in a similar experiment (Mitra and Denny, 1993). That measure is based on the linear regression of the expected on the observed values where we look not only at the magnitude of the correlation coefficient *R*, but also at the values of the slope *m* and the intercept *h*. Under ideal conditions, both the correlation as well as the slope should be equal to one, and the intercept should be equal to zero. For the three life tables the values of all these measures may be seen in Table 2.

Table 2. Estimated Values of the Indices e, R, m and h for the Three Life Tables.

		Index			
Country	*Model*	*e*	*R*	*m*	*h*
Botswana	Brass	.08375	.99635	.96550	.02527
	Two-Standard	.05776	.99894	1.00208	-.00143
Bahrain	Brass	.03488	.99890	1.01540	-.00822
	Two-Standard	.01118	.99993	.99762	.00165
Japan	Brass	.04497	.99887	.96411	.02953
	Two-Standard	.02050	.99925	.97701	.01876

From the figures and Table 2, it is obvious that the two standard model fits the life tables quite well and performs better than the logit model of Brass. While the quality of the logit model has the disadvantage of its being influenced by the choice of the standard, the two-standard model appears free from such defect since the standards can be chosen to cover the entire range of variation of the life expectancy.

Summary and Concluding Remarks

The limits set by the boundary condition on the parameters of Brass's model involving the logits of the life table survivorship function $\ell(x)$ led us to the development of an alternative model. This model, based on a linear relationship of the reciprocals of the $\ell(x)$ function of a given table with the same of two other appropriately chosen standard life tables, has shown encouraging results. For improving the quality of fit we have chosen the method of nonlinear regression to estimate the expected values rather than inverting the expected values of the reciprocals of $\ell(x)$ from its linear regression equation. It is certainly possible that Brass's model subjected to similar treatment will produce a better fit but was not explored since it failed to meet the boundary condition.

We would conclude by noting that the three life tables chosen for this experiment show a tendency for both the parameters c and d to be positive. Also, both parameters seem to decline with increase in life expectancy of the given life table. Further experiment with a large number of life tables will reveal the patterns of variation in these two parameters from a historical as well as from a global perspective.

Acknowledgement

I sincerely appreciate the assistance of Clark Denny, a doctoral student in our department, in generating the figures and the tables of this paper.

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Received, September 1994; revised, February 1994.