Canadian Studies in Population 38, No. 1–2 (Spring/Summer 2011):169–77.

A comment on Thomas K. Burch's paper "Does demography need differential equations?"

Frans J.C. Willekens
Honorary Fellow, Netherlands Interdisciplinary Demographic Institute
Emeritus Professor of Population Studies, University of Groningen
willekens@nidi.nl

Introduction

Thomas Burch addresses an issue that is key for the vitality of our discipline: our ability to deal with (complex) dynamic systems. A population is a dynamic system and most will agree that it is complex if we consider population composition in some detail, the drivers of change, the many feedback mechanisms, and the consequences of demographic changes. Burch raises two questions. First, why has demography made relatively little use of differential equations to model dynamic systems? Second, why has demography made so little use of modern software for modeling dynamic systems? These questions subsume other questions. Does demography keep up with developments in natural and social sciences? Are demographers sufficiently familiar with the analytical strategies and technologies designed to investigate complex processes? In his view, demography, and in particular general and social demography, lag behind. The remedy is training.

The core business of demography is the study of population processes. Populations change because children are born and persons migrate and die. If fewer children are born and people live longer, an ageing population is the sure outcome. The ageing of a population may be postponed in the short run by attracting young migrants, and in the long run by increasing the birth rate. Immigration changes the composition of the population, and that may change its identity. Even without migration, populations change when old cohorts are replaced by new cohorts. Cohort replacement is a main mechanism of social change. If fertility decline should be stopped and reversed, conditions must be created that make raising children attractive. They include adequate work-life balance with flexibility and security. The processes that drive population change are many. They are intertwined and embedded in a changing context. Population processes are complex indeed. Most dynamic systems that really matter are complex. The climate system and the financial system are two cases in point. Are demographers aware of the analytical strategies and technologies that are available today to represent and investigate complex systems? Should they use these tools to study demographic change? The differential equation is the major analytical strategy to investigate processes of change, and systems dynamics software a major technology to solve the equations.

Burch asserts that demography would be a stronger discipline if it had assimilated the regular use of differential equations in general, and systems dynamics software in particular—not for data crunching but to develop theories on demographic change. It

would help demographers think better about complex social and demographic processes and develop new theories about how population systems evolve. It would clarify how the processes of differentiation and selection result in population diversity, and how it is influenced by interaction with the social, cultural, and physical environment. He rightly stresses that differential equations are theoretical models which describe how demographic processes work. They are no substitute for statistical modeling and inference. Differential equations have parameters, the values of which may depend on several factors. That dependence structure must be captured in a statistical model, and the parameters of that model must be estimated from data. I agree with Burch on the potential of differential equations in the study of population processes. I also agree that the field would be stronger if processes are represented in continuous time using differential equations, although I could very well live with colleagues who represent processes in discrete time using difference equations. In this commentary I show that demographers use differential equations more than they admit. The equations remain often implicit. Relatively simple extensions of commonly used differential equations capture properties of population dynamics that are known to exist but are considered too difficult to study.

Technology supports science in the quest for new theories and new knowledge from data. It is no different in demography. Most demographers rely on modern statistical software to identify patterns in data and to explain behaviour at a stage of life in terms of characteristics at the time of the behaviour, conditions at earlier stages, and characteristics of the household, the community, and the welfare state in which persons live. They are able to identify spurious dependencies that are results of unobserved characteristics and selection. The use of mathematical software in demography did not keep pace with the use of statistical software. Mathematical software such as Mathematica and Matlab is as easy to use as Stata or SAS. I agree with Burch that training provides the solution.

The structure of this commentary is as follows. In the next section I discuss differential equations and show that they are more common in demography than is generally accepted, and that they can be used to develop theories of population dynamics. The integration of mathematical modeling with differential equations, and statistical modeling with regression equations, is illustrated. In the following section I briefly discuss software. That discussion leads to a concrete proposal: to use a software package or computer language that facilitates both statistical and mathematical modeling. That language is R, a free software environment for statistical computing and graphics that has utilities to solve differential equations. Section 4 is the conclusion.

Differential equations

Differential equations describe quantities that vary continuously in time. They are equations of motion. In ordinary differential equations, the change in the variable of interest (the state variable) is a function of the value of that variable. The function has at least one parameter that needs to be estimated from empirical observations. The Malthusian or exponential growth model is essentially an ordinary differential equation describing population change. It is $dP(t)/dt = rP(t)$, where $P(t)$ is the population size at time t, dt is a small time interval, and the parameter of the equation is the rate of change r, assumed to be constant. The equation is solved by expressing $P(t)$ in terms of the population size at a previous point in time and the rate of change. The solution is the well-known expo-

nential model $P(t) = \exp[rt]P(0)$, where $P(0)$ is the population size at baseline. In demography, the solution is often written as the recursive equation $P(t+1) = \exp[r]P(t)$.

The differential equation is a process model. It describes a process in continuous time. It is the main mathematical technique in the study of dynamical systems. As stressed by Burch, differential equations are theoretical models. They represent our understanding of how a system works and evolves. In many applications, the state variable is not the variable of interest. The variable of interest is a quantity that depends on the state variable. That quantity and the state variable may be affected by an intervention or some exogenous factors. The differential equation may be augmented by (1) an equation that relates the variable of interest to the state variable; and (2) a term that represents the external influence. The model that results is the state-space model

$$\frac{dP(t)}{dt} = aP(t) + b\,u(t)$$
$$y(t) = cP(t) + du(t)$$

where $u(t)$ is the exogenous input and $y(t)$ the output. The parameters are a, b, c, and d. The parameters may vary in time (time-varying system), and the state variable, the input variable, and the output variable may be vectors representing the structure of a system in addition to its size. The state-space model is a uniform mathematical format across disciplines to investigate dynamical systems. It would indeed make demography stronger if that uniform format was adopted in the study of population dynamics. In demography, the input variable may be immigration and the output variable the size (and structure) of the labour force.

The life table and the projection model as we know them are process models that describe changes in cohort size or population structure. These changes may be expressed in continuous time by differential equations, or in discrete time by difference equations. Difference equations are used more often than differential equations. In the life table, the size of a birth cohort at a given age is expressed in terms of the size at a previous age and the number of deaths during the age interval. Let x denote age and n the length of the interval. The cohort size at age $x+n$ is the cohort size at age x minus the deaths during the interval from x to $x+n$. The equation, shown in any textbook, e.g., Preston et al. (2001: 59), is $l(x+n) = l(x) - {}_nd(x)$, where $l(x)$ is the cohort size at age x and ${}_nd(x)$ is the number of deaths during the interval $(x, x+n)$. The number of deaths during the interval depends on the rate of death and the person-years of exposure to the risk of dying: ${}_nd(x) = {}_nm(x)\,{}_nL(x)$, where ${}_nm(x)$ is the death rate during the interval and ${}_nL(x)$ is the exposure time in person-years. Although in the model the death rate varies from one interval to the next, the risk level varies continuously with age. In other words, the process takes place in continuous time, even when we describe the process in discrete time (time intervals). A description in discrete time is generally dictated by the data. We usually do not have exact ages at death, but only have ages in completed years or age groups of five years. The theoretical model of cohort size is a model in continuous time. The theoretical model results when the time interval is very small, infinitesimally small. Such a small age interval is generally denoted by dx, and the death rate during the small interval by instantaneous death rate or force of mortality. The instantaneous death rate may be viewed as the death rate ${}_nm(x)$ when the interval tends to zero:

$$\mu(x) = \lim_{n \to 0} {}_nm(x),$$

where $\mu(x)$ is the instantaneous death rate at age x (i.e., during the interval from x to $x + dx$). The theoretical model of cohort size is a differential equation, $dl(x)/dx = -\mu(x)\, l(x)$, with $dl(x)$ the change in cohort size during the small interval dx. The equation is the most simple differential equation used in demography and is presented, although implicitly, in most textbooks, including Preston et al. (2001) and Hinde (1998). Introductory texts such as Rowland (2003) do not use differential equations explicitly, but they introduce the distinction between continuous time and discrete time when comparing exponential growth (compounding at every moment) and geometric growth (compounding at fixed intervals). A few texts, such as Namboodiri (1990), adopt a more explicit process perspective on demographic change. The conclusion is warranted that demographic texts avoid the explicit use of differential equations, although differential calculus is applied to explain the concept of instantaneous death rate. It is a small step to introduce differential equations explicitly.

Differential equations facilitate the modeling of more complex phenomena. Consider exponential population growth. Suppose the growth rate depends on population density, defined as the ratio of the population size $P(t)$ and the carrying capacity K. The differential equation is

$$\frac{dP(t)}{dt} = r\left[1 - \frac{P(t)}{K}\right] P(t).$$

The growth rate

$$r^* = r\left[1 - \frac{P(t)}{K}\right]$$

varies in time. The population grows exponentially when it is small relative to the carrying capacity, but the growth rate slows down when the density increases, and it becomes zero when the carrying capacity is reached. The population size never exceeds K. The solution of the differential equation is a logistic function. This simple illustration shows that (1) the logistic model is an extension of the exponential model; and (2) the logistic model incorporates a feedback mechanism. The rate of change is dependent on the outcome of that change. Differential equations are powerful tools for describing population processes. The model may be extended in order to capture more realistic features of the process. Burch mentions an extension which allows that at small population densities, the population goes to extinction. There is widespread evidence of the Allee effect in natural populations and several causal mechanisms have been proposed, the most obvious being the difficulty of finding mates. To extend the logistic model to a model that describes extinction if population is below a threshold, and growth if it exceeds the threshold, the growth rate is multiplied by the term $P(t) - A$, with A the threshold. If the population drops below A, it goes to extinction. The model is still quite simple, but it captures properties of real populations.

Differential equations also facilitate the modeling of interacting populations. The predator-prey model is a case in point. The prey population grows exponentially unless subject to predation. In the presence of predators, the growth rate of prey is suppressed at a degree that is proportional to the number of predators. Predators grow only in the presence of prey. In the absence of prey, their size declines exponentially and they become extinct. The model is a theoretical model that describes the mechanism of change. Because the growth rate of predators depends on prey and vice-versa, the changes in

the predator and prey populations are represented by two differential equations that are solved together as a system of equations:

$$\frac{dP_1(t)}{dt} = P_1(t)\left[\alpha - \beta P_2(t)\right]$$
$$\frac{dP_2(t)}{dt} = -P_2(t)\left[\gamma - \delta P_1(t)\right],$$

where P_1 is the number of prey and P_2 is the number of predators. It is the Lotka-Volterra equation. The parameters represent the interaction of the two species. That interaction is the main subject of study in population biology. It does not receive much attention in demography, although the model is not much more complicated than the exponential growth model. Solving the systems of equations is more complex, but modern software facilitates that task.

Another illustration of interacting populations is a population that consists of subpopulations that exchange people. For instance, in a system of regions, a regional population gains people through births and in-migration, and loses people through deaths and out-migration. The number of in-migrants from a given region depends on the size of the population of the sending region. Consider a population with mobility only. The process is described by a system of two simultaneous equations:

$$\frac{dP_1(t)}{dt} = -\mu_{12} P_1(t) + \mu_{21} P_2(t)$$
$$\frac{dP_2(t)}{dt} = \mu_{12} P_1(t) - \mu_{21} P_2(t) \quad ,$$

where μ_{ij} is the instantaneous rate of migration from region i to region j. The system of equations may be written as a matrix equation:

$$\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \frac{dP_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\mu_{12} & \mu_{21} \\ \mu_{12} & -\mu_{21} \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \mu \ \mathbf{P}(t)$$

The solution of the equation system is $\mathbf{P}(t) = \exp[-\mu t]\ \mathbf{P}(0)$. The system of differential equations describes a continuous-time Markov model. That model is the workhorse of multi-state demography. Fertility and mortality may be introduced by adding birth and death rates to the diagonal elements of μ, and the parameters may be age-specific.

In this section I presented simple differential equations that are used in demography and ecology, the field that deals with interacting species in a common environment. Differential equations are theoretical models. They do not compete with statistical models. They are complementary instead. Statistical models, such as regression models, do not describe mechanisms of change. They describe statistical associations between variables. For instance, a regression analysis may indicate that smokers have a higher death rate than non-smokers, or that persons with a higher education are less likely to suffer cognitive impairments at old age. The analysis does not reveal the causal mechanism that produces that empirical relation. I therefore fully agree with Burch that regression models do not describe how a system works. I disagree that it means that they are uninformative. The detection of a statistical association may lead the way to uncover the underlying causal mechanism. This perspective on statistical modeling as supporting but not replacing causal modeling is particularly relevant in survival analysis and event history analysis dealing

with processes. Blossfeld and Rohwer (2002: 24) make the following statement: "The important task of event history modeling is not to demonstrate causal processes directly, but to establish relevant empirical evidence that can serve as a link in a chain of reasoning about causal mechanisms." For many years we have known that smoking leads to lung cancer and cardiovascular disease, but only recently the negative link between higher education and cognitive impairment was discovered. Higher cognitive reserves resulting from education do not prevent brain damage but suppress its clinical expression. Education improves the mind's resilience to neuropathological damage. For a statistical analysis of the association between education and cognitive impairment at old age, see Reuser et al. (2011). For the description of the mechanism, see Brayne et al. (2010).

Regression models may be usefully combined with differential equations. Consider a differential equation describing the survival process: $dl(x)/dx = -\mu(x)\, l(x)$. The instantaneous death rate varies with age. Suppose we have two subpopulations, and an attribute is present in one and absent in the other. Assume that the effect of the attribute on the mortality rate is the same for all ages. The mortality hazards are proportional, and the Cox proportional hazard model can be used to describe the association between the covariate and the mortality rate by age: $\mu(x) = h(x) \exp[\beta X]$, where X is 0 if the attribute is absent and 1 if it is present. The parameter β measures the effect of the presence of the attribute on the death rate, and $h(x)$ is the baseline hazard, i.e., the death rates by age for those without the attribute (reference category). The solution of the differential equation with the instantaneous death rate replaced by the Cox model is

$$l(x) = \exp\left[-\ \exp(\beta X)\int_0^x h(\tau)\ d\tau\ \right] l(0) \qquad .$$

The differential equation describes the survival process. The regression equation captures the effects of covariates on the parameters of that process. The integration of differential equations and event history models yields a powerful tool for demographic analysis. Tuma and Hannan (1984) were among the first in the social sciences to explicitly integrate differential equations and event history modeling.

Most of the processes demographers are concerned about occur in continuous time. Some processes, such as elections, occur in discrete time and should be described by discrete-time models and difference equations.

Software

Burch sees in appropriate software an opportunity to engage demographers with little mathematical background in process thinking, using differential equations. He mentions special-purpose software tools, such as Dynamo and Stella, and general-purpose software such as Mathematica. Some, such as Mathematica and Matlab, are alive and flourishing. Other products, such as Dynamo, seem to have lost momentum. Some languages are powerful but less known, such as the M language used by Hilderink (2000) to simulate population growth in seventeen regions of the world. For most software tools, Wikipedia is a good initial source of information. For common models, such as the Lotka-Volterra equation, there are Java applets available on the internet.[1] Demographers, like other people, are reluctant to invest in a new computer language. Most stay with the package or language they acquired in college and these are likely to be general purpose

1. See, e.g., http://www.sumanasinc.com/webcontent/animations/content/predatorprey.html (accessed March 30, 2011).

tools for statistical data analysis. SPSS and Stata are popular packages among social scientists, including demographers. R and SAS are popular among statisticians. The first two packages have no facilities to solve differential equations. R has. Unlike SPSS and Stata, R is designed as a high-level programming language and is free (http://www.r-project.org). It is a free software environment for statistical computing and graphics. Thousands of scholars around the world contribute packages and make source code available to the Comprehensive R Archive Network (CRAN; http://cran.r-project.org). It is a unique experiment in international scientific collaboration. R Wiki is dedicated to the collaborative writing of R documentation, and R-Forge offers a central platform for the development of R packages and R-related software. Soetaert et al. (2010a) review the types of differential equations that can be solved with packages contributed to CRAN. Those familiar with R have direct access to software for solving differential equations. They can use the graphics capabilities of R and can easily integrate the packages in a broader analysis. By way of illustration, Box 1 shows the code that solves the logistic model and displays the result. One can easily vary the growth rate (r) and the carrying capacity K. The function *vdpol* specifies the differential equation and the function *ode* solves the equation, using the *deSolve* package contributed by Soetaert et al. (2010b). The R code to solve the Lotka-Volterra equation is shown in Box 2 to illustrate that solving differential equations in R is not much more than (1) writing a function (*LVmod*) that specifies the systems of equations; and (2) specifying parameter values and calling the *ode* function. The R code to simulate a population involving predation and Allee effects is available from this author. The code uses the model and the parameters presented by Duman and Merdan (2009). Duman and Merdan use Matlab.

Box 1. R code to solve the differential equation of the logistic model.

```
library (deSolve)
r <- 0.05
K <- 1000
Pini <- 10
vdpol <- function (t,P,Z)
  { list (r * (1-P/K) * P) }
s <- ode (y=Pini,func=vdpol,time=seq(0,200,by=1),parms=c(r,K))
plot (s,type="l",lwd=2,ylab="P",main="Logistic population growth")
```

Conclusion

Does demography needs differential equations? I agree with Burch that the field would be stronger if differential calculus was part of the curriculum and differential equations were used widely. Differential calculus is the basic mathematical tool for anyone interested in studying change beyond descriptive and statistical analysis. Differential equation models are common in demography but are rarely used explicitly. That should change. When demographers become familiar with differential equation models of simple processes in continuous time, they will want to add realism and, hence, complexity. Among the modern software to solve differential equations, one stands out because it allows both differential equation modeling and statistical modeling. In addition, it is free, has superb graphics capabilities (important for simulation), and the source code is available for inspection. It is R, the high-level programming language for statistical computing and graphics. One advantage is that a number of graduate programs in demography

already teach R. The *ode* function of the *deSolve* package can easily be included paving the way to applications of the predator-prey model and other continuous-time process models in theoretical and applied demographic research. Thomas Burch's paper is likely to trigger new developments in demography that make the field stronger by using analytical strategies and technologies designed to investigate complex processes.

Box 2. R code to solve the predator-prey Lotka-Volterra model.

```
LVmod <- function(Time, State, Pars) {
  with(as.list(c(State, Pars)), {
    Ingestion    <- rIng  * Prey*Predator
    GrowthPrey   <- rGrow * Prey*(1-Prey/K)
    MortPredator <- rMort * Predator
    dPrey        <- GrowthPrey - Ingestion
    dPredator    <- Ingestion*assEff - MortPredator
    return(list(c(dPrey, dPredator)))
  })
}
pars    <- c(rIng   = 0.2,    # /day, rate of ingestion
             rGrow  = 1.0,    # /day, growth rate of prey
             rMort  = 0.2 ,   # /day, mortality rate of predator
             assEff = 0.5,    # -, assimilation efficiency
             K      = 10)     # mmol/m3, carrying capacity
yini    <- c(Prey = 1, Predator = 2)
times   <- seq(0, 200, by = 1)
out     <- ode(func = LVmod, y = yini, parms = pars, times = times)
summary(out)
matplot(out[,1], out[,2:3], type = "l", xlab = "time", ylab = "Conc",
        main = "Lotka-Volterra", lwd = 2)
legend("topright", c("prey", "predator"), col = 1:2, lty = 1:2)
```

Source: Soetaert et al. 2010c, p. 69.

References

Blossfeld, H.-P. and G. Rohwer. 2002. *Techniques of event history modeling: New approaches to causal analysis*. 2nd edn. Mahwah, NJ: Lawrence Erlbaum Associates.

Brayne, C., P.G. Ince, H.A.D. Keage, I.G. McKeith, F.E. Matthews, T. Polvikoski et al. 2010. Education, the brain and dementia: Neuroprotection or compensation? *Brain* 133(8):2210–16.

Duman, O. and H. Merdan. 2009. Stability analysis of continuous population model involving predation and Allee effect. *Chaos, Solitons and Fractals* 41:1218–22.

Hilderink, H. 2000. *World Population in Transition: An Integrated Regional Modeling Framework*. Amsterdam: Thela-Thesis.

Hinde, A. 2005. *Demographic Methods*. London: Arnold.

Namboodiri, N.K. 1990. *Demographic Analysis: A Process Perspective*. San Diego: Academic Press.

Preston, S.H., P. Heuveline, and M. Guillot. 2001. *Demography: Measuring and Modeling Population Processes*. Oxford (UK): Blackwell.

Reuser, M., F.J. Willekens, and L. Bonneux. 2011. Higher education delays and shortens cognitive impairment: A multistate life table analysis of the U.S. Health and Retirement Study. *European Journal of Epidemiology*. DOI: 10.1007/s10654-011-9553-x.

Rowland, D.T. 2003. *Demographic Methods and Concepts*. Oxford (UK): Oxford University Press.

Soetaert, K., T. Petzoldt, and R.W. Setzer. 2010a. Solving differential equations in R. *The R Journal* 2(2):5–15. Available at http://journal.r-project.org (accessed May 18, 2011).

Soetaert, K., T. Petzoldt, and R.W. Setzer. 2010b. Solving differential equations in R: Package deSolve. *Journal of Statistical Software* 33(9):1–25. Available at http://www.jstatsoft.org/v33/i09/paper (accessed May 18, 2011).

Soetaert, K., T. Petzoldt, and R.W. Setzer. 2010c. deSolve [software package]. Available at http://www.ehu.es/ccwmuura/irakaskuntza/sinum/R/deSolve_manual.pdf (accessed May 18, 2011).

Tuma, N. and M. Hannan. 1984. *Social Dynamics: Models and Methods*. New York: Academic Press.