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AGE-PARITY-NUPTIALITY-SPECIFIC STABLE POPULATION MODEL

S. Krishnamoorthy
Population Studies Center, University of Pennsylvania, Philadelphia, U.S.A.

Résumé — Les modèles âge-parité-nuptialité, développés jusqu'ici par les démographes avaient une restriction quelconque sur l'état matrimonial ou sur l'incidence des naissances dans les états matrimoniaux différents. Cette étude présente une méthode pour l'établissement d'une table de mortalité incorporant l'état matrimonial. On a aussi présenté la méthode de projection de la distribution démographique par âge-parité-nuptialité. Le modèle proposé a été appliqué à la population féminine des Etats Unis pour l'année 1970.

Abstract — The age-parity-nuptiality models so far developed by demographers had one or more restrictions on marital status or on the occurrence of births in different marital states. This paper presents a method of constructing a life table incorporating marital status and parity, and develops the age-parity-nuptiality-specific stable population model recognizing all types of marital status, and births to women in any marital status. A method of projecting age-parity-nuptiality-specific distribution of population has also been presented. The proposed model is applied to the United States female population for the year 1970.

Key Words — **life table, stable population, increment-decrement life table**

I. Introduction

The age-differentiated stable population has dominated the field of mathematical demography since the time of Lotka. Since then numerous attempts have been made by various authors to refine Lotka's model by introducing two important demographic factors, parity and marital status (Welpton, 1946; Keyfitz, 1968; Oechsli, 1975; Das Gupta, 1976). But all of them had one or more restrictions on marital status and/or parity. Most of these models either did not differentiate currently married status from widowed or divorced states or did not allow births to occur to single women. Hoem (1970) has developed a probabilistic fertility model of the life table type allowing births to women in any marital status, but has neither applied his model to any population nor explored the scope of his model's application to stable populations.

This paper attempts to present a method of construction of a life table incorporating marital status and parity, and to develop the age-parity-nuptiality-specific stable population model recognizing all types of marital status and births to women in any marital status. The theoretical advantage of this approach lies particularly in the development of intrinsic rates of birth and growth derivable from the age-parity-nuptiality model. The model presented here is so general that with very slight modification it can be adapted to project population by labor force status, school enrollment, *etc.*

II. Marital Status and Parity Life Table

Designate a state space of mutually exclusive states of marital status and parity. Let us assume that we have k marital states: single, married, divorced, etc.; and s parity states: 0, 1, 2, . . . $s-1$, where $s-1$ represents the parity order of $s-1$ and above. The combination of marital states and parities would produce ks mutually exclusive states: $S_0, S_1, \ldots, S_{s-1}, M_0, M_1, \ldots, M_{s-1}, D_0, D_1, \ldots, D_{s-1}$, *etc.*, and let us designate them by states 1, 2, 3, . . ., ks. Now the difference between the fraction of persons surviving to age x and to age $x + \Delta x$ in state i is given by:

$$\overset{(i)}{l}(x+\Delta x) - \overset{(i)}{l}(x) = \sum_{j=1}^{ks} {}_jM_i(x)\, \overset{(j)}{l}(x)\, \Delta x \tag{1}$$

where ${}_jM_i(x)$ is the instantaneous rate of transition at age x from state j to i, when $i \neq j$, and ${}_iM_i$ is the instantaneous rate of going out of state i due to death and transition to other states, multiplied by minus one. When transition for some specific j to i is impossible, ${}_jM_i = 0$. Dividing both sides by Δx and letting $\Delta x \rightarrow 0$, we get a differential equation.

Hence we have a differential equation for each one of $i = 1, 2, \ldots, ks$. The set of ks simultaneous differential equations can be compactly written in matrix notation as

$$\frac{d\{l(x)\}}{dx} = \underset{\sim}{M}(x)\, \{l(x)\} \tag{2}$$

where $\{l(x)\}$ is column vector consisting of elements $l(x)$ in the ith position, and $\underset{\sim}{M}(x)$ is a $ks \times ks$ matrix whose element in the ith row and jth column is ${}_jM_i(x)$, which is as defined earlier.

Now if we consider $\{l(0)\}$, consisting of unity in the first position and zeros elsewhere — that is, at age zero there is only one individual; she is single and of parity zero — then the probability of her appearance in each of the ks states at age x can be obtained from

$$\{l(x)\} = \Omega_0^x \{l(0)\} \tag{3}$$

where Ω_0^x can be computed as described below.

If we subdivide the interval $(0,x)$ into n segments by means of $(n\text{-}1)$ intermediate points such that each one of the intervals is very small, and choose a point from each sub-interval, say $x_1, x_2, \ldots, x_n$, and designate the width of these sub-intervals by $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, then the solution for Ω_0^x by the Infinitesimal Calculus of Volterra (Gantmacher, 1959) is:

$$\Omega_0^x = exp[\underset{\sim}{M}(x_n)\Delta x_n]\, exp[\underset{\sim}{M}(x_{n-1})\Delta x_{n-1}] \ldots exp[\underset{\sim}{M}(x_1)\Delta x_1] \tag{4}$$

The exponential of a matrix is defined by $\exp(A) = \underset{\sim}{I} + > \underset{\sim}{A} + \underset{\sim}{A}^2/2! + \underset{\sim}{A}^3/3! + \ldots$, where $\underset{\sim}{A}$ is a square matrix and I is a unit matrix. The series always converges causing no problem for the solution.

The number of persons alive in the hypothetical stationary population between age x and $x+5$ is

$$\{{}_5L_x\} = \int_0^5 \{l(x+t)\}\, dt, \tag{5}$$

which can be computed approximately by

$$\{{}_5L_x\} = \frac{5}{2}\, [\, \{l(x+5)\} + \{l(x)\}] \tag{6}$$

For computation of other multi-status life table values we refer the reader to Rogers and Ledent (1976) and Krishnamoorthy (1979).

III. Stable Population

A. Integral Equation Approach

Once we compute the values of $\{l(x)\}$ from (3) and (4), we can make use of it in Lotka's integral equation. The number of female births $B(t)$ at time t can be expressed as

$$B(t) = \int_0^{\omega} \theta\, B(t-x)\ \{l(x)\}'\ \{m(x)\}\ dx, \tag{7}$$

where θ is the probability that a birth will be a girl baby, $\{l(x)\}'$ is the transpose of the column vector $\{l(x)\}$, $\{m(x)\}$ is a column vector of parity- and nuptiality-specific instantaneous force of fertility, and ω is the largest age to which any one lives. Substituting the trial solution $B(t) = Q\ e^{rt}$ in (7), it can be seen that the intrinsic rate of increase r is the only real root of the integral equation (Keyfitz, 1968: p. 100).

$$\Psi(r) = \int_0^{\omega} e^{-rt}\ \theta\ \{l(x)\}'\ \{m(x)\}\ dx = 1 \tag{8}$$

In order to evaluate the value of the intrinsic rate of increase from the equation (8), we write it in discrete form using five year age groups as

$$\Psi(r) = \sum_{x=\alpha/5}^{\beta/5-1} \frac{\theta\ e^{-(5x+2.5)r}}{l_0}\ \{_5L_{5x}\}'\ \{m(5x+2.5)\} = 1 \tag{9}$$

where $\{_5L_{5x}\}'$ is the row vector consisting of the person-years lived in different states within the age interval $5x$ to $5(x+1)$ derived from the multiple status life table and l_0 is the radix of the life table. For an iterative solution of (9), the reader is referred to Keyfitz (1968:111-112, 138).

Let $c(x, i)$ be the stable age-parity-nuptiality distribution, that is, the proportion of women of age x and in state i. The distribution of $c(x, i)$ and the intrinsic birth rate b are given by

$$b = 1/[\int_0^{\omega} e^{-rx}\{l(x)\}'\ \{1\}\ dx] \tag{10}$$

and

$$c(x, i) = b\ e^{-rx}\ l^{(i)}(x) \tag{11}$$

where $\{1\}$ is the column vector consisting of unity everywhere.

Once we have $c(x, i)$ and the basic rates in the model, it is possible to compute many stable population parameters like age distribution of women in different states, intrinsic first marriage rates, intrinsic rates of widowhood and divorce, of births of specific order, legitimate and illegitimate birth rates, etc., including intrinsic net and gross reproduction rates.

B. Projection Approach

Stable population analysis can be carried out by the projection approach. With slight modification in the life table construction described earlier we get the L_x values to be used in the projection matrix. Rogers' (1975) multiregional projection process could then be adapted with slight rearrangement of elements of age-parity-nuptiality-specific birth rates.

The survivorship $\{_5L_x\}$ described earlier is not directly useful for constructing the projection matrix, since it is not possible to derive the conditional survivorship ratios from it. For the projection matrix we need to know the probability for an individual aged x to $x+5$ in state i to be in the age range $x+5$ to $x+10$ exactly after five years. To compute this, let us approximate this quantity to the probability that an individual in state i at exact age $x+5/2$ will be in state j at exact age $x+15/2$. Now using the property of matrizer, sometimes known as matricant or matrizant (Gantmacher, 1959: Chapter 14), we can show that

$$\underset{\sim}{S}(x) \simeq \Omega_{x+5/2}^{x+15/2} \tag{12}$$

where the element in jth row of ith column of the matrix, $\underset{\sim}{S}(x)$, is the probability that an individual in state i at age $x+5/2$ will be in state j at age $x+15/2$.

Instead of this crude approximation to $\underset{\sim}{S}(x)$, let us try one possible improvement. First let us write the theoretical relationship between $\underset{\sim}{\ell}(x)$, Ω and $\underset{\sim}{S}(x)$.

$$\underset{\sim}{S}(x) = \int_0^5 \Omega_{x+t}^{x+5+t} \underset{\sim}{\ell}(x+t)\, dt \left[\int_0^5 \underset{\sim}{\ell}(x+t)\, dt\right]^{-1} \tag{13}$$

where $\underset{\sim}{\ell}(x)$ is a diagonal matrix and $\int_0^5 \underset{\sim}{\ell}(x+t)\, dt$ is assumed to be nonsingular. The superscript -1 stands for the inverse. Writing (14) differently

$$\underset{\sim}{S}(x) = \int_{-5/2}^{5/2} \Omega_{x+5/2+t}^{x+15/2+t} \underset{\sim}{\ell}(x+5/2+t)\, dt \left[\int_0^5 \underset{\sim}{\ell}(x+t)\, dt\right]^{-1} \tag{14}$$

Expanding Ω and $\underset{\sim}{\ell}$ inside the first integral in Taylor's series and taking only the first two terms (that is assuming the elements of both Ω and $\underset{\sim}{\ell}$ are all linear) in the expansion and multiplying the two series inside the first integral and integrating, we get

$$\underset{\sim}{S}(x) = \Omega_{x+5/2}^{x+15/2} + \frac{5^3}{12} \left[\Omega_{x+5/2}^{x+15/2}\right]' \underset{\sim}{\ell}'(x+5/2)\, {}_5\underset{\sim}{L}_x^{-1} \tag{15}$$

where the superscript $(')$ stands for the first derivative and

$${}_5\underset{\sim}{L}_x = \int_0^5 \underset{\sim}{\ell}(x+t)\, dt.$$

Now substituting

$$\left[\Omega_{x+5/2}^{x+15/2}\right]' \simeq \frac{1}{2\times 5} \left[\Omega_{x+5}^{x+10} - \Omega_{x-5}^{x}\right]$$

and

$$\underset{\sim}{\ell}'(x+5/2) \simeq -\frac{1}{2\times 5^2} \left[{}_5\underset{\sim}{L}_{x-5} - {}_5\underset{\sim}{L}_{x+5}\right]$$

in (15), we get a better approximation for $\underset{\sim}{S}(x)$ as

$$\underset{\sim}{S}(x) \simeq \Omega_{x+5/2}^{x+15/2} + \frac{1}{48} \left[\Omega_{x+5}^{x+10} - \Omega_{x-5}^{x}\right] \left[{}_5\underset{\sim}{L}_{x+5} - {}_5\underset{\sim}{L}_{x-5}\right] {}_5\underset{\sim}{L}_x^{-1} \tag{16}$$

Now the elements of $\underset{\sim}{S}(x)$ can be used to fill in the elements of the growth matrix except for the first ks rows. The improvement in the approximation in (16) depends on the assumption of linearity of the elements of both Ω and $\underset{\sim}{\ell}$. This is unlikely but including higher order terms in the analysis will improve our results when this assumption is too optimistic. However, for the youngest and oldest age groups the above method is not applicable.

The projection matrix is not complete without an estimate of surviving births during the time of one step projection. Denote the age-state-specific annual female birth rate by ${}_5F_x^{(i)}$ for the age group x to $x+5$ in the state i. Let $\underset{\sim}{F}(x)$ be a $ks \times ks$ matrix consisting of the elements ${}_5F_x^{(i)}$ in the ith position of the first row and the rest of the elements being zeros. Define

$$\underset{\sim}{B}(x) = \frac{5^{L(l)}_{o}}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5)\, \underset{\sim}{S}(x)] \tag{17}$$

Now the matrices $\underset{\sim}{B}(x)$ and $\underset{\sim}{S}(x)$ can be arranged to form, what Feeney (1970) calls, the generalized Leslie Matrix.

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{B}(\alpha-5) & \underset{\sim}{B}(\alpha & \cdots\cdots\cdots & \underset{\sim}{B}(\beta-5) & \underset{\sim}{O} \cdot\cdot \underset{\sim}{O} & \underset{\sim}{O} \\ \underset{\sim}{S}(0) & \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{O} & \cdots\cdots\cdots & \underset{\sim}{O} & \underset{\sim}{O} \cdot\cdot \underset{\sim}{O} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{S}(5) & \underset{\sim}{O} & \underset{\sim}{O} & \cdots\cdots\cdots & \underset{\sim}{O} & \underset{\sim}{O} \cdot\cdot \underset{\sim}{O} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{S}(10) & \underset{\sim}{O} & \cdots\cdots\cdots & \underset{\sim}{O} & \underset{\sim}{O} \cdot\cdot \underset{\sim}{O} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{S}(15) & \cdots\cdots\cdots & \underset{\sim}{O} & \underset{\sim}{O} \cdot\cdot \underset{\sim}{O} & \underset{\sim}{O} \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots\cdots & \cdot & \cdot \;\cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots\cdots & \cdot & \cdot \;\cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots\cdots & \cdot & \cdot \;\cdots & \cdot \\ \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{O} & \underset{\sim}{O} & \cdots\cdots\cdots & \underset{\sim}{O} & \underset{\sim}{O} \cdot\cdot \underset{\sim}{S}_{(z-5)} & \underset{\sim}{O} \end{bmatrix}$$

The matrix expression of the multiple status growth process is:

$$\{K^{t+5}\} = \underset{\sim}{H}\{K^{(t)}\} \tag{18}$$

where

$$\{K^{(t)}\} = \begin{Bmatrix} \{K^{(t)}_{(0)}\} \\ \{K^{(t)}_{(5)}\} \\ \{K^{(t)}_{(10)}\} \\ \cdot \\ \cdot \\ \cdot \\ \{K^{(t)}_{(z)}\} \end{Bmatrix}$$

and $\{K^{(t)}_{(x)}\}$ is a column vector consisting of ks elements, where the ith element is the population at time t in state i in the age range x to $x+5$. It is worth noting the arrangement of elements in the generalized Leslie matrix. In our case the survivors of new born first enter the state one, which represents single with parity zero. This assumption is valid for almost all modern populations; for some traditional societies, like seventeenth century India, where child marriage was prevalent, some changes are needed in the construction of $\underset{\sim}{B}(x)$ matrices.

IV. Some Remarks

It can be recognized that in our model, by suitable definition of state space, the specificity of either marital status or parity or both can be eliminated or any other characteristic can be included. Thus our model is more general, simple enough to comprehend, and permits simple and general computer program. The only limitation is in the computing cost, which is slightly higher, and this limitation is insignificant when compared to its other benefits.

In our model we have assumed independence of rates of occurrences of events. One way of taking care of such situations is to introduce duration-dependence in the force of transition by including duration of stay in the current state as one of the variables in defining the state space.

V. *Application to United States Data, 1970*

Though our model is general enough to take into account as many marital states as are conceivable, we have taken for the present application only two states, single and ever-married. This restriction is mainly due to lack of availability of data. Table 1 presents the age-parity-nuptiality-specific fertility rates and age-specific first marriage rates. In the absence of first marriage rates of single women by parity, we assume that

TABLE 1 MARRIAGE RATES AND FERTILITY RATES FOR SINGLE WOMEN AND FERTILITY RATES FOR EVER-MARRIED WOMEN: UNITED STATES, 1970

Age group	Marital status*	Marriage rates per 1,000	Fertility rates per 1000 by parity i							
			0	1	2	3	4	5	6	7+
15 - 17.5	S	36.6	16.0	127.8	116.8	←		45.2		→
	M	-	779.2	133.6	58.8	41.9	29.3	←	23.1	→
17.5- 20	S	154.6	26.5	121.0	132.5	←		63.2		→
	M	-	578.3	219.5	174.0	151.3	83.6	←	75.7	→
15 - 20	S	88.2	20.5	123.2	129.5	←		61.4		→
	M	-	614.8	205.9	156.3	131.5	74.7	←	69.9	→
20 - 25	S	237.1	24.1	108.2	159.5	←		171.5		→
	M	-	305.5	247.7	154.3	159.9	156.7	152.4	136.6	102.0
25 - 30	S	129.2	11.9	49.9	78.4	←		146.6		→
	M	-	216.7	239.1	123.0	109.0	114.1	130.8	143.4	154.3
30 - 35	S	60.6	5.3	22.1	35.2	←		92.5		→
	M	-	91.5	120.1	69.3	61.3	63.9	75.9	91.7	123.0
35 - 40	S	38.5	2.4	8.5	12.4	←		47.2		→
	M	-	28.4	36.5	23.9	25.4	30.3	38.2	50.5	84.5
40 - 45	S	22.2	0.6	1.5	2.4	←		12.2		→
	M	-	4.6	5.6	4.2	5.7	8.0	11.7	16.4	33.5
45 - 50	S	14.5	0.1	0.2	0.2	←		0.7		→
	M	-	0.3	0.2	0.2	0.3	0.4	0.9	1.2	3.3

* S = Single; M = Ever-married
SOURCE: Das Gupta, Prithwis. (1976)

first marriage rates depend only on age and not on parity of single women. Owing to the same problem of lack of data, we also assume the age-specific death rates are the same for all marital statuses and parity; though the model can accept different rates.

Application of equation (3) and (4) yield the increment-decrement life table (Table 2) which gives the probability at birth of a girl child's being of a given marital status and parity at age x under the United States female, 1970 forces of death, marriage, and fertility. Application of equation (6) yields Table 3 which shows expected person-years lived in each age, marital status and parity group.

The stable population analysis by integral equations (8) and (9) provides an intrinsic growth rate of 5.95 per 1000 women. By applying equation (10) we get the intrinsic birth rate of 16.77 per 1000 women. Stable age-parity-nuptiality distribution of 10,000 female population is computed using equation (11) and is given in Table 4.

Das Gupta (1976) gives a crude growth rate of 9.31 per 1000 women, an intrinsic

TABLE 2 PROBABILITY AT BIRTH THAT A WOMAN WILL BE IN A GIVEN MARITAL STATUS AND PARITY AT AGE X: UNITED STATES, 1970

Age	Marital Status*	Parity 0	1	2	3	4	5	6	7+	Total
0	S	1.000	0.0	0.0	0.0	0.0	-	-	-	1.000
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	S	0.979	0.0	0.0	0.0	0.0	-	-	-	0.979
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	S	0.977	0.0	0.0	0.0	0.0	-	-	-	0.977
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	S	0.975	0.0	0.0	0.0	0.0	-	-	-	0.975
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	S	0.543	0.047	0.011	0.002	0.000	-	-	-	0.603
	M	0.139	0.166	0.053	0.010	0.001	0.0	0.0	0.0	0.369
25	S	0.146	0.023	0.009	0.004	0.002	-	-	-	0.184
	M	0.186	0.249	0.225	0.088	0.028	0.007	0.001	0.000	0.785
30	S	0.072	0.013	0.006	0.002	0.003	-	-	-	0.096
	M	0.101	0.164	0.304	0.185	0.076	0.026	0.008	0.003	0.869
35	S	0.051	0.010	0.005	0.002	0.003	-	-	-	0.070
	M	0.078	0.123	0.287	0.224	0.110	0.042	0.016	0.009	0.888
40	S	0.042	0.008	0.004	0.001	0.003	-	-	-	0.058
	M	0.076	0.113	0.273	0.227	0.121	0.051	0.020	0.013	0.893
45	S	0.037	0.007	0.003	0.001	0.002	-	-	-	0.051
	M	0.077	0.111	0.267	0.224	0.121	0.052	0.021	0.014	0.886
50	S	0.033	0.007	0.003	0.001	0.002	-	-	-	0.046
	M	0.078	0.109	0.262	0.219	0.118	0.051	0.021	0.014	0.871
55	S	0.032	0.007	0.003	0.001	0.002	-	-	-	0.045
	M	0.076	0.106	0.254	0.212	0.115	0.049	0.020	0.014	0.845
60	S	0.031	0.006	0.003	0.001	0.002	-	-	-	0.043
	M	0.072	0.101	0.242	0.203	0.110	0.047	0.019	0.013	0.807
65	S	0.029	0.006	0.003	0.001	0.002	-	-	-	0.040
	M	0.068	0.094	0.227	0.190	0.103	0.044	0.018	0.012	0.756
70	S	0.026	0.005	0.002	0.001	0.002	-	-	-	0.036
	M	0.061	0.085	0.205	0.171	0.093	0.040	0.016	0.011	0.682
75	S	0.022	0.004	0.002	0.001	0.001	-	-	-	0.031
	M	0.052	0.073	0.174	0.146	0.079	0.034	0.014	0.010	0.580
80	S	0.017	0.003	0.002	0.001	0.001	-	-	-	0.024
	M	0.040	0.055	0.133	0.111	0.060	0.026	0.010	0.007	0.443
85	S	0.011	0.002	0.001	0.000	0.001	-	-	-	0.015
	M	0.026	0.036	0.086	0.072	0.039	0.017	0.007	0.005	0.286

*S = Single; M = Ever-married

growth rate of 5.74 in Lotka's age model and 6.75 from his age-parity-nuptiality model. The present model using almost the same set of data yielded a rate of 5.95. The difference between Das Gupta's results and ours could be due to the assumptions inherent in these two models and the approximations involved in computation. It is also worth noting that there is not much difference between the intrinsic rates obtained by the Lotka

TABLE 3 PERSON-YEARS LIVED IN EACH AGE-PARITY-MARITAL STATUS GROUP BY A FEMALE CHILD JUST BORN: UNITED STATES, 1970

Age group	Marital status*	Parity 0	1	2	3	4	5	6	7+	Total
0–	S	4.911	0.0	0.0	0.0	0.0	–	–	–	4.911
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5–	S	4.889	0.0	0.0	0.0	0.0	–	–	–	4.889
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10–	S	4.881	0.0	0.0	0.0	0.0	–	–	–	4.881
	M	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15–	S	3.993	0.142	0.024	0.004	0.000	–	–	–	4.163
	M	0.325	0.301	0.069	0.010	0.001	0.000	0.0	0.0	0.706
20–	S	1.521	0.175	0.054	0.017	0.006	–	–	–	1.773
	M	1.009	1.081	0.703	0.217	0.057	0.012	0.002	0.000	3.080
25–	S	0.525	0.088	0.036	0.015	0.012	–	–	–	0.676
	M	0.700	1.032	1.384	0.686	0.250	0.076	0.021	0.007	4.157
30–	S	0.306	0.057	0.026	0.010	0.013	–	–	–	0.412
	M	0.448	0.712	1.485	1.031	0.464	0.168	0.059	0.029	4.396
35–	S	0.232	0.045	0.021	0.008	0.013	–	–	–	0.319
	M	0.386	0.588	1.399	1.130	0.577	0.232	0.088	0.054	4.454
40–	S	0.195	0.039	0.018	0.006	0.012	–	–	–	0.270
	M	0.382	0.559	1.350	1.127	0.603	0.256	0.102	0.069	4.448
45–	S	0.175	0.035	0.016	0.005	0.011	–	–	–	0.242
	M	0.388	0.549	1.322	1.106	0.598	0.256	0.104	0.072	4.394
50–	S	0.164	0.033	0.015	0.005	0.010	–	–	–	0.228
	M	0.384	0.536	1.288	1.078	0.583	0.249	0.101	0.070	4.290
55–	S	0.158	0.032	0.015	0.005	0.010	–	–	–	0.219
	M	0.369	0.516	1.240	1.038	0.561	0.240	0.097	0.068	4.129
60–	S	0.149	0.030	0.014	0.005	0.009	–	–	–	0.207
	M	0.349	0.488	1.173	0.982	0.531	0.227	0.092	0.064	3.906
65–	S	0.137	0.028	0.013	0.004	0.009	–	–	–	0.191
	M	0.321	0.449	1.079	0.903	0.488	0.209	0.085	0.059	3.593
70–	S	0.120	0.024	0.011	0.004	0.008	–	–	–	0.167
	M	0.282	0.394	0.946	0.792	0.428	0.183	0.074	0.052	3.150
75–	S	0.097	0.020	0.009	0.003	0.006	–	–	–	0.135
	M	0.228	0.318	0.764	0.639	0.346	0.148	0.060	0.042	2.544
80–	S	0.069	0.014	0.006	0.002	0.004	–	–	–	0.095
	M	0.161	0.224	0.539	0.451	0.244	0.104	0.042	0.029	1.796
85+	S	0.070	0.014	0.007	0.002	0.004	–	–	–	0.098
	M	0.165	0.230	0.553	0.463	0.251	0.107	0.043	0.030	1.843

*S = Single; M = Ever-married.

model and the age-parity-nuptiality model, but there is a large difference between crude rate and the intrinsic rate in the Lotka model. This result implies that the observed age distribution of women is far from the stable age distribution. It also suggests that the distribution of women within a given age group according to marital status and parity does not differ greatly between the observed and the stable population, or that the differences

TABLE 4 STABLE AGE-PARITY-MARITAL STATUS DISTRIBUTION PER 10,000 FEMALE POPULATION: UNITED STATES, 1970

Age group	Marital status*	Parity 0	1	2	3	4	5	6	7+	Total
0–	S	811	0	0	0	0	-	-	-	811
	M	0	0	0	0	0	0	0	0	0
5–	S	784	0	0	0	0	-	-	-	784
	M	0	0	0	0	0	0	0	0	0
10–	S	760	0	0	0	0	-	-	-	760
	M	0	0	0	0	0	0	0	0	0
15–	S	603	21	4	1	0	-	-	-	629
	M	49	45	10	2	0	0	0	0	107
20–	S	223	26	8	3	1	-	-	-	260
	M	148	159	103	32	8	2	0	0	452
25–	S	75	13	5	2	2	-	-	-	96
	M	100	147	197	98	36	11	3	1	592
30–	S	42	8	4	1	2	-	-	-	57
	M	62	98	205	143	64	23	8	4	608
35–	S	31	6	3	1	2	-	-	-	43
	M	52	79	188	152	77	31	12	7	597
40–	S	25	5	2	1	2	-	-	-	35
	M	50	73	176	147	79	33	13	9	579
45–	S	22	4	2	1	1	-	-	-	31
	M	49	69	167	140	76	32	13	9	555
50–	S	20	4	2	1	1	-	-	-	28
	M	47	66	158	132	72	31	12	9	526
55–	S	19	4	2	1	1	-	-	-	26
	M	44	61	148	124	67	29	12	8	492
60–	S	17	3	2	1	1	-	-	-	24
	M	40	56	136	113	61	26	11	7	452
65–	S	15	3	1	0	1	-	-	-	21
	M	36	50	121	101	55	23	10	7	403
70–	S	13	3	1	0	1	-	-	-	18
	M	31	43	103	86	47	20	8	6	343
75–	S	10	2	1	0	1	-	-	-	14
	M	24	34	81	68	37	16	6	4	269
80–	S	7	1	1	0	0	-	-	-	10
	M	16	23	55	46	25	11	4	3	184
85+	S	7	1	1	0	0	-	-	-	10
	M	16	23	55	46	25	11	4	3	184

* S = Single; M = Ever-married-

are compensatory in nature in terms of their effect on the intrinsic growth rate. A perusal of the observed and the stable distributions by age, parity and nuptiality supports the former.

The projection approach was also used for the stable population analysis described earlier. Equation (16) was tried for the improved conditional survivorship estimates for

TABLE 5 PROJECTED AGE-PARITY-NUPTIALITY DISTRIBUTION OF UNITED STATES FEMALE POPULATION FOR THE YEAR 2,000 (FIGURES IN 1000'S)

Age	Marital	Parity								Total
group	status*	0	1	2	3	4	5	6	7+	
0–	S	10607	0	0	0	0	–	–	–	10607
	M	0	0	0	0	0	0	0	0	0
5–	S	10343	0	0	0	0	–	–	–	10343
	M	0	0	0	0	0	0	0	0	0
10–	S	10501	0	0	0	0	–	–	–	10501
	M	0	0	0	0	0	0	0	0	0
15–	S	9025	327	54	6	0	–	–	–	9412
	M	418	657	110	6	0	0	0	0	1191
20–	S	2956	361	126	40	7	–	–	–	3490
	M	1241	3173	1625	483	109	12	1	0	6644
25–	S	983	164	71	32	19	–	–	–	1269
	M	1112	1923	2702	1427	557	162	39	9	7931
30–	S	526	98	45	19	22	–	–	–	710
	M	680	1159	2513	1823	873	330	113	48	7539
35–	S	466	91	42	16	25	–	–	–	640
	M	696	1113	2743	2298	1232	517	201	114	8914
40–	S	410	81	38	14	24	–	–	–	567
	M	725	1103	2759	2385	1335	589	240	155	9291
45–	S	344	67	30	11	19	–	–	–	471
	M	694	1018	2526	2177	1223	543	226	154	8561
50–	S	319	54	21	7	14	–	–	–	415
	M	737	1013	2198	1743	941	415	178	142	7367
55–	S	207	31	14	7	15	–	–	–	274
	M	491	673	1625	1440	852	410	191	182	5864
60–	S	169	20	10	7	14	–	–	–	220
	M	360	451	1123	1149	793	439	221	254	4790
65–	S	161	15	8	5	12	–	–	–	201
	M	333	409	945	976	714	420	222	306	4325
70–	S	175	13	6	5	9	–	–	–	208
	M	372	473	966	884	603	346	188	285	4117
75–	S	162	9	5	3	6	–	–	–	185
	M	367	484	874	688	429	241	127	204	3414
80–	S	143	5	2	2	3	–	–	–	155
	M	309	379	590	410	239	130	69	120	2246
85+	S	147	5	3	2	3	–	–	–	160
	M	410	425	532	337	193	111	63	124	2195
Total										134220

* S = Single; M = Ever-married.

five-year age groups, with unsatisfactory results. This is suggestive of the limitation in our assumption of linearity of the elements of Ω and $\underset{\sim}{\ell}$. This difficulty could have been overcome by taking the higher order terms in the Taylor's expansion of equation (14). However, for the present application we used the crude approximation given by (12). The projected population in the year 2000 is given in Table 5 as an illustration.

The intrinsic rate of growth could also be obtained from the dominant eigenvalue of the projection matrix H. The dominant eigenvalue for the present data is computed by using the power method (Rogers, 1971:Chapter 7) and it is $\lambda = e^{5r} = 1.02646$, which gives the intrinsic rate of growth of 5.22 per 1000 women. The difference between this and the one obtained by the application of the characteristic equation (8) is small and it is due to the crude approximation in estimating the conditional survivorship used in the projection matrix. Further work is indicated to develop a method of computing more accurate conditional survivorship probabilities.

In conclusion, it should be pointed out that the reliability of the numerical results depends on the reliability of the data used. The main thrust of this paper is to provide mathematical models that can be conveniently used for routine application rather than provision of reliable estimates.

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