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# Using the probabilistic fertility table to test the statistical significance of fertility trends

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**Abstract**

At below replacement level, fertility changes are subtle and complex; and distinguishing statistically significant trends from random shifts is becoming a relevant issue. The probabilistic fertility table describes the uncertainty of the childbearing process, and provides a significance test for the annual changes of various fertility measures, which is essential for distinguishing between a statistically significant change from a random fluctuation. This paper provides an analytical model for the total fertility of the probabilistic fertility table, and extends the significance test to period trends that include multiple annual changes. The extended significance test indicates that complex annual changes could accumulate to become a significant trend. Applying the analytical model and extended test to the total fertility of Canada, it indicates that the 2000–11 upward trend is statistically significant and, therefore, supports recently projected future increases of total fertility.

**Keywords:** childbearing uncertainty, probabilistic fertility table, fertility trend, significance test, Canada.

**Résumé**

En-dessous du seuil de remplacement des générations, les changements à la fertilité sont subtils et complexes. Aussi, il est devenu pertinent de pouvoir distinguer les tendances significatives au plan statistique des écarts aléatoires. Le tableau probabiliste de fertilité décrit l'incertitude liée au processus de reproduction et fournit un critère de signification des changements annuels dans les diverses mesures de fertilité, élément essentiel pour distinguer un changement important au plan statistique des fluctuations aléatoires. Cet article fournit un modèle analytique pour l'ensemble du tableau probabiliste de fertilité et élargit la portée de cette mesure aux tendances dans le temps incluant les multiples changements annuels. Ce critère élargi indique que les changements complexes annuels peuvent représenter une tendance significative. En appliquant le modèle analytique et le critère au tableau de fertilité du Canada, on constate que la tendance à la hausse de 2000–11 est importante au plan statistique et, par conséquent, augure des hausses futures dans la fertilité totale.

**Mots-clés :** incertitude relative à la procréation, tableau probabiliste de fertilité, tendance en fertilité, critère de signification.

## Introduction

Total fertility in more developed regions declined across replacement level in the middle of the 1970s, and has stayed below that level since then (United Nations 2015). This unprecedented phenomenon caused the 'Low Fertility Trap' hypothesis (Lutz, Skirbekk, and Testa 2006). Starting from year 2000, many developed countries experienced slight increases in their fertility rates (Goldstein, Sobotka, and Jasilioniene 2009; Bongaarts and Sobotka 2012). After 2008, however, some developed countries have seen minor declines in their fertility levels (Goldstein et al. 2013). At above replacement level, remarkable annual declines in total fertility often demon-strate obvious downward trends. At levels below replacement, however, fertility changes are much more subtle and complex. As a result, whether the successive annual changes in a certain period compose a genuine trend or a random shift is becoming an important question.

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To provide a statistical answer to this question, analysis is needed of the uncertainty of the childbearing process. The conventional total fertility (TF) is defined as the sum of age-specific fertility rates that do not distinguish the order of births and the parity of women, and therefore do not offer a basis to carry out probabilistic analysis. Standard errors of TF are estimated, for example, in the Demographic and Health Surveys (e.g., Statistics Indonesia et al. 2013) to measure sampling errors. These standard errors, however, do not indicate the uncertainty of the childbearing process for which the data collected from a whole country are available.

In order to investigate the uncertainty of the childbearing process, an analytic model for the total fertility of the probabilistic fertility table[2] is proposed in this paper. Based on this analytic model, the statistical significance test is extended from an individual annual change to a period trend that includes multiple annual changes. Furthermore, the period trend significance test indicates that multiple insignificant and complex annual changes could gradually accumulate to a significant trend, thus providing important insight for the analysis of fertility change and for fertility projections.

## An analytic model of total fertility

The probabilistic fertility table describes the uncertain childbearing process of a hypothetical cohort of women who are subject to neither to mortality nor migration. Following the rationale of developing probabilistic life tables (Li 2015a), the number of women in the hypothetical cohort is specified to minimize the differences between the hypothetical cohort and the observed population. Let the minimal and maximal reproductive age be $a_{\min}$ and $a_{\max}$,[3] respectively; and let the number of women at age $a_{\min}$ be $l_0(a_{\min})$, where subscript 0 refers to having zero children. Then, the number of women is $l_0(a_{\min})$ at all reproductive ages, because of the absence of mortality and migration. Denote the number of observed female population at age a by $f_p(a)$. Then, minimizing $\sum_{a=a_{\min}}^{a_{\max}}[l_0(a_{\min}) - f_p(a)]^2$ leads to $l_0(a_{\min})$ being the average of the observed female population over reproductive ages:

$$l_0(a_{\min}) = \frac{\sum_{a=a_{\min}}^{a_{\max}} f_p(a)}{a_{\max} - a_{\min} + 1}. \tag{1}$$

The uncertain childbearing process can be simulated by assuming that a woman delivers children independently from the others according to the Bernoulli distribution and the probability of delivering children by age and parity[4] in a certain year, starting from the minimal reproductive age $a_{\min}$. With a value of $l_0(a_{\min})$, this simulation can be repeated for each of the $l_0(a_{\min})$ women; and a sample of the childbearing process of the hypothetical cohort is obtained, which provides a sample fertility table. A large number of sample fertility tables then comprises a probabilistic fertility table (Li 2015b).

Focusing on total fertility, an analytical model can be derived as below. Let the number of children of the *jth* woman at age $a_{\max}$ be a random variable, $X_j$. Then, the total fertility of the probabilistic fertility table $(TF_f)$, which is defined as the average number of children per woman at age $a_{\max}$, is

$$TF_f = \frac{\sum_{j=1}^{l_0(a_{\min})} X_j}{l_0(a_{\min})}. \tag{2}$$

Further, let the probability for a woman to have $i$ children at age $a_{\max}$ be $p_i$. Then, the mean and variance of $X_j$ are $\sum_i i \cdot p_i$ and $\sum_i (i - \sum_k k \cdot p_k)^2 \cdot p_i$ . According to (2), the mean and variance of $TF_f$ are therefore

2. The quantitative values of conventional total fertility are often close to that of the probabilistic fertility table total fertility.
3. In this paper, $a_{min}$ and $a_{max}$ are taken as the commonly employed 15 and 50 years, respectively.
4. The birth parity of a woman refers to the number of children she has delivered.

$$\mu = Mean(TF_f) = \sum_i i \cdot p_i \tag{3}$$

and

$$\sigma^2 = Var(TF_f) = \frac{\sum_i (i - \sum_k k \cdot p_k)^2 \cdot p_i}{l_0(a_{\min})} \tag{4}$$

Formulas (3) and (4) indicate that the mean of $TF_f$ is independent from $l_0(a_{\min})$, but the variance of $TF_f$ is inversely proportional to $l_0(a_{\min})$. In other words, the uncertainty of $TF_f$ is smaller when the population size is larger, and vice versa.

Finally, when $\mu$ is not close to zero and $l_0(a_{\min})$ is larger than 30 (see Agresti and Finlay 1997: 104), the law of large numbers provides an analytical model for $TF_f$ as

$$TF_f \sim N(\mu, \sigma^2)\,. \tag{5}$$

Using observed data, the mean and variance of $TF_f$ are estimated according to the formulas in the appendix, as $\hat{\mu}$ and $\hat{\sigma}^2$.

## Significance tests of the changes and trends of $TF_f$

### The statistical significance of an annual change of $TF_f$

When total fertilities are forecasted by time-series models, they are correlated over time because of containing the same modelling errors of previous years. It is worth noting that the uncertainty is introduced from the errors of modelling the over-time changes in total fertility. Moreover, in a time-series model, uncertainty cannot be assigned to total fertility in the initial years, because there are no modelling errors.

On the other hand, in the probabilistic fertility table, $TF_f$ is uncertain in any year. The uncertainty does not come from modelling errors but from the uncertain childbearing process of the hypothetical cohort, in which a woman's childbearing behaviour is assumed to be independent from the others. Subsequently, the uncertainties in $TF_f$(t) and $TF_f$(t+1) are caused by the uncertain childbearing processes of two hypothetical cohorts. Since each woman's childbearing behaviour is assumed to be independent from the others, the childbearing processes of the two hypothetical cohorts are consequently independent.

Let the mean and variance of $TF_f(t)$ and $TF_f(t+1)$ be $\mu(t)$, $\mu(t+1)$, $\sigma^2(t)$, and $\sigma^2(t+1)$, respectively. Then, setting the null hypothesis as

$$H_0: \quad \mu(t) = \mu(t+1)\,, \tag{6}$$

and noting that $TF_f(t)$ and $TF_f(t+1)$ are independent, and that $l_0(a_{\min})$ is large (so $\hat{\sigma}^2$ is close to $\sigma^2$), we obtain

$$Z(t) = -\frac{TF_f(t) - TF_f(t+1)}{\sqrt{\hat{\sigma}^2(t) + \hat{\sigma}^2(t+1)}} \sim N(0,1)\,, \tag{7}$$

where a negative sign is used to make a positive *Z,* representing an increase in total fertility.

If the estimated value of $Z(t)$, namely,

$$\hat{z}(t) = -\frac{\hat{\mu}(t) - \hat{\mu}(t+1)}{\sqrt{\hat{\sigma}^2(t) + \hat{\sigma}^2(t+1)}}\,, \tag{8}$$

is found to be outside (−1.96, 1.96), which occurs with a probability smaller than 0.05 according to the null hypothesis, then the null hypothesis is rejected, implying that the change in $TF_f$ is statistically significant. Otherwise, the change in $TF_f$ cannot be concluded to be statistically significant. The above procedure can also be used to test the significance of the difference between the total fertility in two separate years or of two populations, and in general could be called a two-point significance test.

## The statistical significance of a period trend of $TF_f$

For a period that includes multiple years, namely from year 1 through year $t$, the significance of the difference in $TF_f$ between year 1 and year $t$ can be tested using the above two-point significance test. When the difference between year 1 and year $t$ is insignificant, there is no significant trend in period $[1,t]$, because a significant trend should not lead to an insignificant difference. When the difference between year 1 and year $t$ is significant, however, the trend in period $[1,t]$ may not necessarily be significant. A simple example is that the annual change between year 1 and year 2 is just significant, and there is no change later. In this example, then, the difference between year 1 and year $t$ is significant, because the distance between the two points does not matter in a two-point significance test. But intuitively there is no significant trend when $t$ is large, because among the multiple annual changes only the first one is just significant and all others are zero. In real situations, there may be significant and insignificant annual changes over a certain time interval. These changes may not be exactly zero and they may cancel each other. In these situations, whether there is a significant trend depends on the details of the annual changes.

To test the significance of a period trend that includes multiple annual changes, the difficulty is that in calculating the average change over a period, the middle values of $TF_f$ will cancel each other and only the first and last values matter. A solution to overcome this difficulty is to construct the average of odd and even ranked changes. Here, *odd rank* signifies that the earlier year of each annual change being an odd number; and *even rank* is defined analogously. Subsequently, the odd and even ranked average changes, namely $Y_1$ and $Y_2$, are constructed as:

$$\begin{aligned} Y_1 &= -\frac{[TF_f(1)-TF_f(2)]+[TF_f(3)-TF_f(4)]+...+[TF_f(t_1-1)-TF_f(t_1)]}{t_1/2}, \\ Y_2 &= -\frac{[TF_f(2)-TF_f(3)]+[TF_f(4)-TF_f(5)]+...+[TF_f(t_2-1)-TF_f(t_2)]}{(t_2-1)/2}. \end{aligned} \tag{9}$$

Noting that $TF_f(i)$ and $TF_f(j)$ are independent and that $l_0(a_{\min})$ is large, we obtain the following relations for the variances:

$$Var(Y_1) = \frac{\sum_{i=1}^{t_1}\sigma^2(t)}{t_1^2/4} \approx \frac{\sum_{i=1}^{t_1}\hat{\sigma}^2(t)}{t_1^2/4}, \quad Var(Y_2) = \frac{\sum_{i=2}^{t_2}\sigma^2(t)}{(t_2-1)^2/4} \approx \frac{\sum_{i=2}^{t_2}\hat{\sigma}^2(t)}{(t_2-1)^2/4}. \tag{10}$$

To test the statistical significance of period trends, the null hypothesis can be set as *no trend*,

$$H_0: \quad Mean(Y_1) = Mean(Y_2) = 0, \tag{11}$$

and the alternative hypothesis can be set, for even and odd ranked trends both exist and do not cancel each other, as

$$H_a: \quad Mean(Y_1) > 0, \quad Mean(Y_2) > 0, \quad or \quad Mean(Y_1) < 0, \quad Mean(Y_2) < 0. \tag{12}$$

According to the null hypothesis, there are

$$Z_1 = \frac{Y_1}{\sqrt{Var(Y_1)}} = -\frac{[TF_f(1) - TF_f(2)] + ... + [TF_f(t_1 - 1) - TF_f(t_1)]}{\sqrt{\sum_{t=1}^{t_1} \hat{\sigma}^2(t)}} \sim N(0,1),$$
$$Z_2 = \frac{Y_2}{\sqrt{Var(Y_2)}} = -\frac{[TF_f(2) - TF_f(3)] + ... + [TF_f(t_2 - 1) - TF_f(t_2)]}{\sqrt{\sum_{t=2}^{t_2} \sigma^2(t)}} \sim N(0,1). \tag{13}$$

Subsequently, the corresponding sample values of $Z_1$ and $Z_2$ are

$$\hat{z}_1 = -\frac{[\hat{\mu}(1) - \hat{\mu}(2)] + ... + [\hat{\mu}(t_1 - 1) - \hat{\mu}(t_1)]}{\sqrt{\sum_{t=1}^{t_1} \hat{\sigma}^2(t)}},$$
$$\hat{z}_2 = -\frac{[\hat{\mu}(2) - \hat{\mu}(3)] + ... + [\hat{\mu}(t_2 - 1) - \hat{\mu}(t_2)]}{\sqrt{\sum_{t=2}^{t_2} \hat{\sigma}^2(t)}}. \tag{14}$$

Then, if

$$\hat{z}_1 \geq 1.96, \hat{z}_2 \geq 1.96, \text{ or } \hat{z}_1 \leq -1.96, \hat{z}_2 \leq -1.96, \tag{15}$$

the null hypothesis is rejected and the alternative hypothesis is in favour, which indicates that both the even and odd ranked trends are statistically significant and they do not cancel each other. In other words, the whole trend is statistically significant. On the other hand, if (15) does not stand, then the whole trend cannot be concluded to be statistically significant, although the null hypothesis could still be rejected.[5]

## Multiple insignificant and complex changes could accumulate to a significant trend

Consider now the formulas in (14). Compared to the differentials that enlarge the effect of random fluctuation in the numerators, the over-time changes in $\hat{\sigma}(t)$ in the denominators are negligible in common situations (e.g., Figure 3). Thus, $\hat{\sigma}(t)$ can be approximately replaced by their average, $\overline{\sigma}$. Subsequently, denoting the over-time average of odd ranked $\hat{z}(t)$ by $\overline{z}_1$, the following relations hold:

$$\hat{z}_1 = -\frac{[\hat{\mu}(1) - \hat{\mu}(2)] + ... + [\hat{\mu}(t_1 - 1) - \hat{\mu}(t_1)]}{\sqrt{\sum_{t=1}^{t_1} \hat{\sigma}^2(t)}} \approx -\frac{[\hat{\mu}(1) - \hat{\mu}(2)] + ... + [\hat{\mu}(t_1 - 1) - \hat{\mu}(t_1)]}{\overline{\sigma}\sqrt{t_1}}$$
$$= \frac{\sqrt{2}}{\sqrt{t_1}} \cdot \{-\frac{[\hat{\mu}(1) - \hat{\mu}(2)]}{\overline{\sigma}\sqrt{2}} - ... - \frac{[\hat{\mu}(t_1 - 1) - \hat{\mu}(t_1)]}{\overline{\sigma}\sqrt{2}}\} \tag{16}$$
$$\approx \frac{\sqrt{t_1}}{\sqrt{2}} \cdot \{\frac{\hat{z}(1) + ... + \hat{z}(t_1 - 1)}{t_1 / 2}\} = \sqrt{\frac{t_1}{2}} \cdot \overline{z}_1, \quad t_1 \geq 4.$$

For the same reason,

$$\hat{z}_2 \approx \sqrt{\frac{t_2 - 1}{2}} \cdot \overline{z}_2, \quad t_2 \geq 5, \tag{17}$$

where $\overline{z}_2$ is the over-time average of the even-ranked $\hat{z}(t)$.

5. Similar to a one-sided significance test ($H_0 : \mu = 0$, $H_a : \mu > 0$), here the null and alternative hypotheses are not complementary; rejecting the null hypothesis does not lead to accepting the alternative hypothesis.

Now, consider a case in which the annual change of $TF_f$ (t) is linear and insignificant ($-1.96<\hat{z}(t)=\overline{z}_1=\overline{z}_2<1.96$). In this case, all individual changes are insignificant, but the period trend is significant when the number of years is large, because (16) and (17) indicate that large values of $t_1$ and $t_2$ will make $\hat{z}_1,\hat{z}_2\geq 1.96$ or $\hat{z}_1,\hat{z}_2\leq -1.96$. In real situations, the changes in $TF_f(t)$ are not linear; $\hat{z}(t)$ are not constant and could be positive in one year but negative in another. Nonetheless, (16) and (17) still indicate that even if individual annual changes are insignificant and contain cancellations over time, $\overline{z}_1$ and $\overline{z}_2$ could still have the same sign, and therefore the period trend could still be significant when the number of years is large.

Although insignificant and complex changes could accumulate to a significant trend, it is not guaranteed. Using (16) and (17), a condition for a trend to be significant is obtained as

$$\overline{z}_1 \geq \frac{1.96}{\sqrt{t_1/2}}, \quad \overline{z}_2 \geq \frac{1.96}{\sqrt{(t_2-1)/2}}, \quad or \quad \overline{z}_1 \leq \frac{-1.96}{\sqrt{t_1/2}}, \quad \overline{z}_2 \leq \frac{-1.96}{\sqrt{(t_2-1)/2}}. \tag{18}$$

Different from the two-point significance test, in which the distance between the two points does not matter, in (18) the length of the period ($t_1$ or $t_2$) matters: the trend is more likely significant when the length of the period is longer, given the values of $\overline{z}_1$ and $\overline{z}_2$.

## Applications

### An analytical model

Based on the law of large numbers, (5) indicates that the probability distribution of $TF_f$ is approximately normal, of which the mean and variance are estimated according to the formulas in the appendix. Using the latest (year 2011) data on age-parity-specific fertility rates of Canada in the Human Fertility Database (MPIDR and VID 2013), the normal distribution of $TF_f$ is computed and shown by the solid curve in Figure 1. Compared to the numerical distribution that is computed through simulation using 1,000 sample fertility tables (Li 2015b) and described by the squares in Figure 1, we see that the analytical model works well.

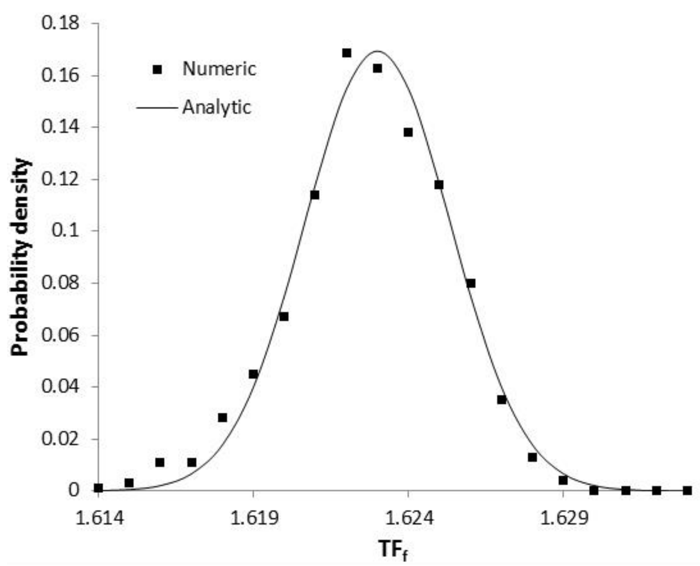


***Figure 1. Probability distribution of total fertility, Canada 2011.***

### Changes and trends in total fertility

Since the year 2000, total fertility has started to increase in many low-fertility countries (Goldstein, Sobotka, and Jasilioniene 2009). Bongaarts and Sobotka (2012) explained the main reasons as the consequence of pro-family policies and a diminishing pace of the postponement of childbearing. After 2008, some developed countries have seen minor declines in their fertility levels, which may be caused by the global financial crisis (Goldstein et al. 2013). These changes appeared also in Canada, as can be seen in Figure 2, in which the values of fertility table total fertility $TF_f$ and conventional total fertility $TF$ are compiled from the Human Fertility Database up to 2011.

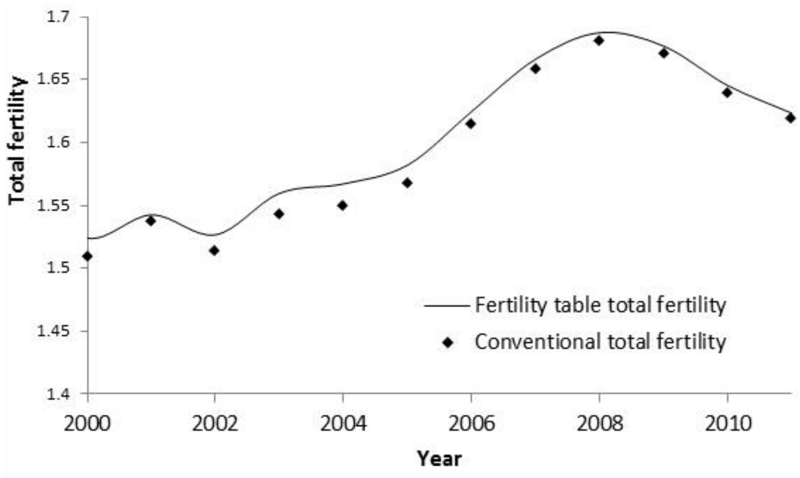


*Figure 2. Total fertility of Canada, 2000–11.*

The differences between the values of $TF_f$ and $TF$ are negligible. The reason could be that the age distributions of woman by birth parity approached that of the hypothetical cohort. Regardless of using $TF_f$ or $TF$, the changes in Canadian total fertility are typical among developed countries: increased slightly from 2000 or 2002; and started to decline after 2008. Starting from 2000 or some later years, the overall trend is an increasing one. The overall increase trend has been used as the basis of fertility and population projections by many low-fertility countries, including Canada (Bohnert et al. 2015). Because these trends are subtle and include offsets, whether they are statistically significant becomes an important question; and an answer for Canada is provided below.

## Results of significance test

Applying (14) to the data for Canada in 2000–11, the results are $\hat{z}_1 = 9.4$ and $\hat{z}_2 = 3.3$, both greater than 1.96. Thus, the overall upward trend in 2000–11 is statistically significant. Because projections are believed to be better when based on a longer period, it is not practically useful to test the trends in shorter periods starting later than 2000. Given the annual declines in total fertility, especially 2008–11, how can the overall upward trend in 2000–11 be statistically significant? It can be explained using Figure 3 and the condition in (18).

In general, complex annual changes that include offsets could accumulate to a statistically significant trend under certain conditions. In the common situation that the standard deviation of $TF_f$ is approximately constant compared to the annual changes in $TF_f$, a simple condition for complex annual changes to compose a statistically significant trend is found as (18).

Although the changes in total fertility are declines in 2001–02 and 2008–11, they are increases in 7 other years. Figure 3 indicates that Canada is in the common situation so that condition (18) applies. Note also that with

$\bar{z}_1 = 3.7 > \frac{1.96}{\sqrt{t_1/2}} = \frac{1.96}{\sqrt{9/2}}$ and $\bar{z}_2 = 2 > \frac{1.96}{\sqrt{(t_2-1)/2}} = \frac{1.96}{\sqrt{9/2}}$, (18) is satisfied and indicates that after offsetting with the declines, the increases still accumulate to a statistically significant trend. Nonetheless, if annual declines after 2011 occurred dominantly, $\bar{z}_1$ and $\bar{z}_2$ would be reduced to break (18), and the annual changes starting from 2000 would not accumulate to a statistically significant trend.

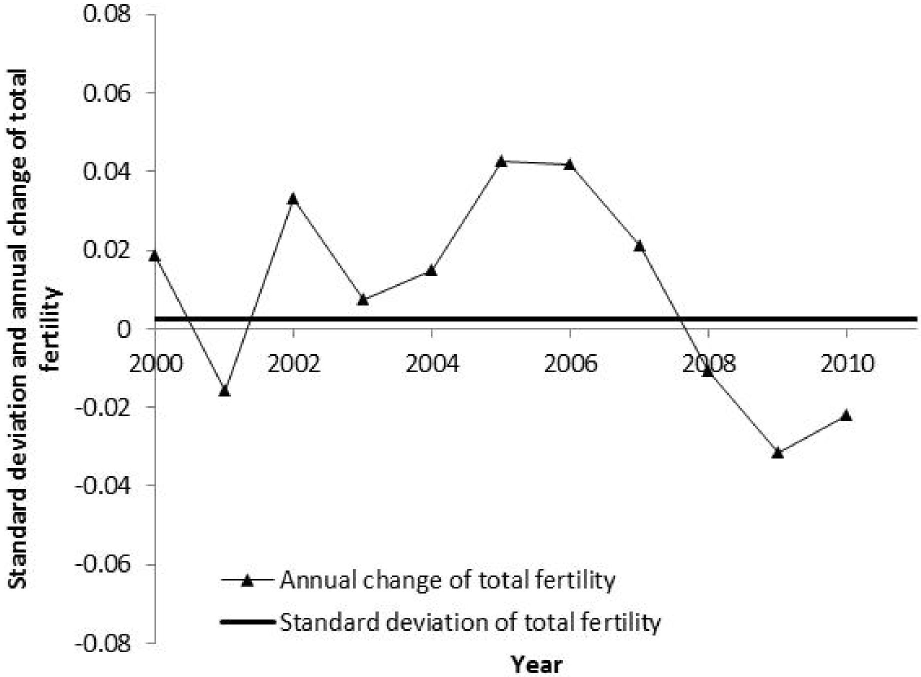


***Figure 3. Standard deviation and annual change of total fertility, Canada 2000–11.***

At below replacement level, the annual changes in total fertility are subtle and complex; and whether they compose a genuine trend or a random shift is an important question. This question is common among low-fertility countries, among which Canada is not an exception: of the 11 recent annual changes, 7 are increases and 4 are declines. Differing from the other countries for which this question remains open, in Canada we see that the 11 recent annual changes have accumulated to a genuine trend.

Basing fertility projections on the trend of a recent period is a common practice. This basis is obviously sounder when the period is longer, and should be more reasonable when the trend is statistically significant. An 11-year increase of trend in total fertility is proper to empirically support the 10-year increase of total fertility in the medium projections of Statistics Canada (Bohnert et al. 2015). Confirming the statistical significance suggests that this 11-year upward trend for Canada is genuine, and provides statistical support to these medium projections.

## Summary

The probabilistic fertility table describes the uncertainty of the childbearing process, and hence provides significance tests for an annual change in various fertility variables. On the other hand, the probabilistic fertility table requires immense calculation. Moreover, how to test the statistical significance of a fertility trend that includes multiple annual changes is still an open question. The purposes of this paper have been to simplify the application of the probabilistic fertility table, and to extend the test of significance from an annual change to a period trend that includes multiple annual changes.

Using the law of large numbers, the total fertility of the probabilistic fertility table is found to obey the normal distribution approximately, whose mean and variance can be estimated using analytical formulas. This analytical model substantially simplifies significance tests, in both description and calculation. It should be mentioned that in the probabilistic fertility table, variables other than total fertility may not obey normal distribution, or may not be described by analytical distributions.

Constructing the even and odd ranked average changes in total fertility over a period, the obstacle of the middle values of $TF_f$ offsetting each other is avoided. Using the normal distribution of total fertility, the even and odd ranked average changes are found to also obey normal distributions. Thus, the null hypothesis that the mean values of the average changes are zero, or there is no trend over the period, can be tested. Furthermore, the test procedure indicates that multiple insignificant and complex annual changes may accumulate to a significant period trend.

Finally, applying the analytical model and extended test to the fertility data of Canada, the results indicate that the 11 recent annual changes in total fertility, of which 7 are increases and 4 are declines, accumulated to a statistically significant trend. This 11-year statistically significant trend supports Statistics Canada's recently projected future increases of total fertility.

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## Appendix

To simplify the equations in this appendix, all the variables are used to represent the corresponding estimated values. For the hypothetical cohort, let the number of women having $(i-1)$ children at age $a$ be $l_{i-1}(a)$, and the number of children delivered by these women at ages $[a,a+1)$ be $b_i(a)$. Then, the probability of delivering the *ith* child in age interval $[a,a+1)$, namely, $q_i(a)$, can be defined as

$$q_i(a) = \frac{b_i(a)}{l_{i-1}(a)}, \quad q_{m+}(a) = \frac{b_{m+}(a)}{l_{(m-1)+}(a)}, \tag{A.1}$$

where $(m-1)+$ indicates the open parity of having the $(m-1)th$ and higher-order children. Using definition (A.1), the childbearing process is written as

$$l_{i-1}(a) = \begin{cases} l_{i-1}(a-1)[1-q_i(a-1)], \quad i=1, \\ l_{i-1}(a-1)[1-q_i(a-1)] + l_{i-2}(a-1)q_{i-1}(a-1), \quad 1 < i < m, \end{cases} \tag{A.2}$$
$$l_{(m-1)+}(a) = l_{(m-1)+}(a-1) + b_{(m-1)}(a-1).$$

Using population data of census and estimates, and data on births of vital registrations, the values of the age-parity-specific fertility rate for a certain time interval can be computed as:

$$M_i(a) = \frac{Number\ of\ the\ ith\ births\ delivered\ by\ below\ women}{Person-years\ of\ women\ having\ (i-1)\ children\ at\ ages\ [a,a+1)}. \tag{A.3}$$

In (A.3), both the numerator and the denominator refer to a certain time interval, which may or may not be a calendar year. It should be mentioned that, although the age interval must be one year for fertility table, the time interval to which a fertility table refers can be flexible such as 5 years. This is important for small populations, of which a longer time interval should contain more births and hence make the age-parity-specific fertility rates more robust.

Because fertility may change only slightly in one year interval of age and a moderate time interval, there is approximately (see Preston, Heuveline, and Guillot, 2001)

$$M_i(a) = m_i(a), \tag{A.4}$$

where $m_i(a)$ represents the age-parity-specific fertility rate of the hypothetic cohort, and is defined as

$$m_i(a) = \frac{b_i(a)}{L_{i-1}(a)}, \tag{A.5}$$

where $L_{i-1}(a)$ represents the person-years of the $(i-1)th$ parity in $[a,a+1)$:

$$L_{i-1}(a) = \int_{y=a}^{a+1} l_{i-1}(y)dy, \tag{A.6}$$

and $l_{i-1}(y)$ represents the number of women of parity $(i-1)$ at age $y$.

Using $m_i(a)$, $q_i(a)$, and $l_{(i-1)}(a)$ can be computed. For $i=1$, $L_0(a)$ is the population exposed to the chance of having the first child at ages $[a,a+1)$. For $i>1$, however, $L_{i-1}(a)$ is not the population exposed to the chance of having the *ith* child at ages $[a,a+1)$, because some women entered parity $(i-1)$ by bearing the $(i-1)th$ child at ages $[a,a+1)$ and thence are not exposed to the chance of having the *ith* child within the rest of the calendar year, according to the assumption that a woman can bear at most one birth in one year.

Under the assumption that the births occur evenly in each age interval, both the decline (due to delivering the *ith* child) and the increase (due to delivering the $(i-1)th$ child) of $l_{i-1}(a)$ are linear functions of age. Thus, $l_{i-1}(a)$ changes with $a$ linearly, and therefore

$$L_{i-1}(a) = 0.5 \cdot [l_{i-1}(a) + l_{i-1}(a+1)]. \tag{A.7}$$

For the case of $i=1$, (A.7) leads to

$$m_1(a) = \frac{b_1(a)}{L_0(a)} = \frac{q_1(a) l_0(a)}{0.5 \cdot [l_0(a) + l_0(a+1)]} \ . \tag{A.8}$$

Using the first line of (A.2), (A.7) is rewritten as

$$m_1(a) = \frac{q_1(a)}{0.5 \cdot [1 + l_0(a+1)/l_0(a)]} = \frac{q_1(a)}{0.5 \cdot [1 + (1 - q_1(a))]}, \tag{A.9}$$

which yields

$$q_1(a) = \frac{m_1(a)}{1 + 0.5 \cdot m_1(a)} \ . \tag{A.10}$$

Equation (A.10) is identical to the corresponding formula in life tables, because $L_0(a)$ is the population exposed to the chance of having the first child at ages $[a,a+1)$.

For the cases of $m>i>1$, (A.7) still yields

$$m_i(a) = \frac{b_i(a)}{L_{i-1}(a)} = \frac{q_i(a) l_{i-1}(a)}{0.5 \cdot [l_{i-1}(a) + l_{i-1}(a+1)]}, \tag{A.11}$$

but now the second line of (A.2) applies, and leads to

$$m_i(a) = \frac{b_i(a)}{L_{i-1}(a)} = \frac{q_i(a) l_{i-1}(a)}{0.5 \cdot [l_{i-1}(a) + l_{i-2}(a) q_{i-1}(a) + l_{i-1}(a)(1 - q_i(a))]}. \tag{A.12}$$

Rewriting (A.12), we obtain

$$q_i(a) = \frac{m_i(a)}{1 + 0.5 \cdot m_i(a)} [1 + \frac{0.5 \cdot l_{i-2}(a) q_{i-1}(a)}{l_{i-1}(a)}], \quad i > 1 \ . \tag{A.13}$$

The difference between (A.10) and (A.13) is caused by that, although $L_{i-1}(a)$ is still the person years, it is no longer the population exposed to the chance of having the *ith* child at ages $[a,a+1)$ for $i>1$. This can be explained as below. For $i>1$, the $(i-1)th$ births make $L_{i-1}(a)$ larger than the population exposed to the chance of bearing the *ith* birth, and the $m_i(a)$ smaller, comparing to that of $i=1$. Thus, as a compensation, (A.13) includes an additional term, compared to (A.10). This additional term makes the calculation slightly complicated.

In (A.13), $q_i(a)$ and $l_{i-1}(a)$ are unknown, and can be solved iteratively together with the second line of (A.2):

$$\begin{aligned} l_{i-1}(a) &= l_{i-2}(a-1) q_{i-1}(a-1) + l_{i-1}(a-1)[1 - q_i(a-1)], \\ q_i(a) &= \frac{m_i(a)}{1 + 0.5 \cdot m_i(a)} [1 + \frac{0.5 \cdot l_{i-2}(a) q_{i-1}(a)}{l_{i-1}(a)}]. \end{aligned} \tag{A.14}$$

The iteration starts from $i=2$, of which $q_1(a)$ and $l_0(a)$ for all $a$ are already computed as the result of $i=1$. Using the first line of (A.14), $l_1(a_{min}+1)$ is obtained, because it is known that $q_2(a_{min})=0$, according to the assumption that a woman could deliver only one child in a year. Subsequently, $q_2(a_{min}+1)$ is obtained from the second line of (A.14). When $q_2(a_{min}+1)$ is known, $l_1(a_{min}+2)$ is obtained from the first line of (A.14), and so is $q_2(a_{min}+2)$ from the second line of (A.14). Repeating this process, $q_2(a)$ and $l_1(a)$ for all $a$ are obtained. Now the iteration reaches $i=3$, of which $q_2(a)$, $q_1(a)$, $l_1(a)$, and $l_0(a)$ are already computed. Here, $q_3(a)$ and $l_2(a)$ for all $a$ can be computed in the way similar to that of $i=2$, starting from $q_3(a)=0$ for $a\le(a_{min}+1)$ according to the assumption that a woman could deliver only one child in a year. Repeating the process, $q_1(a)$ and $l_{i-1}(a)$ for all $i\le(m-1)$ are obtained.

For the open parity $q_{m+}(a)$, the assumption that the births occur evenly in each age interval still leads to

$$m_{m+}(a)=\frac{b_{m+}(a)}{L_{(m-1)+}(a)}=\frac{q_{m+}(a)l_{(m-1)+}(a)}{0.5\cdot[l_{(m-1)+}(a)+l_{(m-1)+}(a+1)]}\,. \tag{A.15}$$

Using the third line of (A.2), we obtain

$$m_{m+}(a)=\frac{b_{m+}(a)}{L_{(m-1)+}(a)}=\frac{q_{m+}(a)l_{(m-1)+}(a)}{0.5\cdot[l_{(m-1)+}(a)+l_{(m-1)+}(a)+l_{(m-2)}(a)q_{(m-1)}(a)]}\,, \tag{A.16}$$

which leads to

$$q_{m+}(a)=m_{m+}(a)[1+\frac{0.5\cdot l_{m-2}(a)q_{m-1}(a)}{l_{(m-1)+}(a)}]\,. \tag{A.17}$$

Different from the case of $i<m$, in which $q_i(a)$ are computed iteratively, all $q_{m+}(a)$ can be computed by (A.17); this is because $l_{(m-1)+}(a)$ can be calculated given that the hypothetical cohort subjects neither mortality nor migration:

$$l_{(m-1)+}(a)=l_0(a_{\min})-\sum_{i=0}^{m-2}l_i(a)\,. \tag{A.18}$$

After obtaining $q_1(a)$ and $l_{(i-1)}(a)$, the probabilities of having $i$ children at the maximal reproductive age are obtained as

$$p_i=\frac{l_i(a_{\max})}{l_0(a_{\min})},\quad p_{(m-1)+}=\frac{l_{(m-1)+}(a_{\max})}{l_0(a_{\min})}\,, \tag{A.19}$$

Finally, the mean ($\mu$) and variance ($\sigma^2$) of total fertility are estimated as

$$\hat{\mu}=\sum_{i=1}^{(m-2)}i\cdot p_i+p_{(m-1)+}\cdot[(m-1)+\frac{\sum_{a=a_{\min}}^{a_{\max}}q_{m+}(a)\cdot l_{(m-1)+}(a)}{l_{(m-1)+}(a_{\max})}] \tag{A.20}$$

and

$$\hat{\sigma}^2=\sum_{i=1}^{(m-2)}(i-\mu)^2\cdot p_i+p_{(m-1)+}\cdot\{[(m-1)+\frac{\sum_{a=a_{\min}}^{a_{\max}}q_{m+}(a)\cdot l_{(m-1)+}(a)}{l_{(m-1)+}(a_{\max})}]-\mu\}^2\,, \tag{A.21}$$

respectively.