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The *Kriyākramakarī*'s Integrative Approach to Mathematical Knowledge

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INTRODUCTION

THIS PAPER WILL FOCUS ON a Sanskrit mathematical treatise called *Kriyākramakarī* (edited without translations by Sarma in 1975).[1] This treatise is a commentary on Bhāskara II's twelfth-century *Līlāvatī*, one of the most famous mathematical treatises of the Sanskrit mathematical tradition (see Padmanabha Rao 2015 for a translated edition). The *Līlāvatī* covers all standard areas of Sanskrit arithmetic and geometry, from the most elementary calculations to advanced procedures, up to but excluding algebra and trigonometric tables, which are covered in other works by the same author.

The *Kriyākramakarī* commentary was written by Śaṅkara Vāriyar (fl. 1500–1560), who left it unfinished, and completed by Nārāyaṇa Mahiṣamaṅgalam (ca. 1500–1575).[2] The two authors belong to the so-called Kerala school of astronomy and mathematics, which flourished in Kerala (south west India) from the fourteenth century to the seventeenth century. Their mathematical culture was the most advanced of its time worldwide, prefiguring European calculus with power-series-like expansion for trigonometric functions and the value of π.[3]

1 The title can be translated as something like "the performer of activity in due order," but this literal translation is not very meaningful here. The three words in the title are derived from the root $\sqrt{}$*kṛ*, which is associated with the semantic field of doing, acting, and performing ("create" is an English derivative of this Indo-European root). The title is therefore a sort of pun, centered round the word *krama*, which, in a mathematical, context could mean something like the correct sequence of an algorithmic procedure or calculation rule.

2 For details about the authors see Sarma 1972: 130, 169; Joseph 2009: 21–2.

3 For surveys of Kerala school mathematics see: Sarma 1972; Joseph 2009; Plofker 2009: ch. 7; Puttaswamy 2012: ch. 13; for translated editions see Sarma et al. 2009; Ramasubramanian and Sriram 2010.

The *Kriyākramakarī* is, to risk an anachronistic expression, an encyclopedic work. It follows the verses of the *Līlāvatī*, but, unlike a typical commentary, is not content with an interpretation of the verses and illustrative examples. It complements Bhāskara's verses by related verses drawn from other treatises (named and unnamed – the latter may sometimes be original additions), and expands the scope by including relevant methods and topics that were not covered by Bhāskara (for a summary of the content of the *Kriyākramakarī* and a specification of its most important additions see Appendix A, p. 99). But even more important than its extended scope is the fact that the *Kriyākramakarī* includes detailed justifications (*upapatti* or *yukti*) of most of the methods and statements which it includes, whether Bhāskara's or others'.

I believe that even after Nārāyaṇa's supposed completion of the treatise, the *Kriyākramakarī* should not be considered a complete work – at least not based on the four manuscripts used for its critical edition (Sarma 1975). The manuscripts indicate various substantial gaps in the source material (Sarma 1975: 99, 145, 153,164, 177, 211, 295). These gaps sometimes correspond to what appears to be unfinished treatment of the subject matter, and this is sometimes the case even where no gap is indicated. This may be due to a defective archetype of the manuscripts, but some of the summary verses that conclude the various sections appear in more than one variant, and some summary verses are appended at the end of the treatise, rather than where their subjects are covered, which suggests that Śaṅkara's text itself contained lapses. Nārāyaṇa did complete the treatise in the sense of adding commentary to verses 200–269 of the *Līlāvatī*, but he does not seem to have intervened in the gaps and hiccups left in Śaṅkara's draft.

Several publications treat specific portions of the *Kriyākramakarī*, including some translations of extracts. Among these, one can find the approximation of π (Katz 2007: 481–93), Govindasvāmin's arithmetic rules quoted in the *Kriyākramakarī* (Hayashi 2000), and Citrabhānu's 21 questions (Hayashi and Kusuba 1998; Mallayya 2011; cf. Wagner 2015). Some of the mathematical contents of the book is also covered (without translating the source material as such) in Vrinda (2014), Mallayya (2002), Gupta (1987), Sarasvati Amma (1979), and the general surveys of the Kerala school quoted above.

The purpose of this paper is not to review the treatment of any specific mathematical subject in the *Kriyākramakarī*, but rather to review its general organization of knowledge. Specifically, I will argue that Śaṅkara's presentation of justification or proof is integrative, rather than hierarchical or cumulative. In other words, the purpose of proofs in the *Kriyākramakarī* is, among other things, to connect various different aspects of mathematics, rather than just to convincingly establish or explain mathematical claims by means of previously known claims.

The next section provides general background on proofs in medieval Indian mathematics. This is followed by a section that surveys sources of knowledge used in the proofs of the *Kriyākramakarī*. The subsequent section will present the evidence for the *Kriyākramakarī*'s integrative approach.

1. PROOF IN MEDIEVAL INDIAN MATHEMATICS

SOME (BUT NOT ALL!) HISTORIANS OF MATHEMATICS used to claim that Indian mathematics had no interest in proofs (Sarma et al. 2009: 267–70). Part of the problem was that non-commented Sanskrit mathematical texts are mostly succinct verse summaries of mathematical algorithms and results, which include no justifications. However, these verses are highly elliptic, and can hardly be deciphered without the aid of a qualified teacher or a detailed commentary. The existence of Indian commentaries that include justifications was already made known to English language readers by Whish (1834), and then again by Indian scholars publishing in English since at least the 1940s (Marar and Rajagopal 1944; Rajagopal and Venkataraman 1949), but the news was slow to be taken in. A list of Sanskrit sources, both manuscripts and print editions, that include justifications was made available by Sarma et al. (2009: xxix–xxxii, 294–96).

Within this tradition, the most elaborate justifications that survive come from Kerala, but Sanskrit mathematical justifications do not start there. Indeed, Keller (2012) records Bhāskara I's seventh-century justifications of Āryabhaṭa's procedures. These justifications include what Keller calls "re-interpretation:" setting a given procedure (e.g., a procedure concerning similar triangles) in terms of another procedure (e.g., the rule of three, usually used for commercial calculations). Re-interpretation could also mean breaking down a procedure into instances of known procedures, such as the rule of three and the "Pythagorean Theorem." Another textual practice is "verification:" a problem that uses the solution of a previous inverse problem as its data is presented, and the coincidence of the new problem's solution with the inverse problem's data verifies the procedure under consideration. Finally, cut-and-paste geometry also appears to underlie some re-interpretations of area calculations. All these methods still play a role in the *Kriyākramakarī*.

A more general discussion of proofs, focusing on later Indian mathematics, is available in Narasimha 2007, Raju 2007: ch. 2, Sarma et al. 2009: 267–310 (summarized in Srinivas 2005), Srinivas 2015, and Divakaran 2016. According to these overviews, the goal of medieval Indian mathematical proofs is not to obtain absolute knowledge, but to resolve doubt and confusion, and convince others.[4] To

4 Note that this interpretation is based on considering only a certain kind of canonized mathematical literature, mostly in Sanskrit, and might not be valid for all genres and Indian languages. It may also underplay the geographical and temporal variety in medieval Indian conceptions of mathematical proof.

obtain these goals, one relies on perception (*pratyakṣa*) and inference (*anumāna*), supported by authentic tradition (*śabda*, see Sarma et al. 2009: 286 f.). Proofs are therefore not meant to be part of a purely logical system founded on axioms, but they are not simple empirical generalization from examples either. In Indian mathematics, sense, reason, and authenticated traditions combine to form mathematical proofs just as they do in other Indian scientific contexts. Mathematics is thus not grounded in an exceptional epistemology, when compared to natural sciences.

If we follow this thread, then a proof is not supposed to meet a-priori criteria for absolute truth (as in the classical Greek proof architecture or later logical reconstructions), but only to answer those doubts, misunderstandings and dissents that happen to arise in actual practice. Moreover, according to Raju (2007: ch. 2), Srinivas (2015) and Divakaran (2016), some of the logical traditions most relevant for mathematical proofs are not bivalent, and may therefore allow some forms of contradictions (for instance, regarding fictional entities). Since imaginary, non-observable and idealized entities are part of Indian mathematical-astronomical calculation and reasoning,

> Indian astronomers were sometimes ready to accommodate inexplicable or even seemingly contradictory procedures as component part of their models. (Srinivas 2015: 232)

Despite this lack of interest in an absolute ground, one can find in Sanskrit mathematics statements that might seem to be foundational (but later I will suggest a different interpretation). These statements usually assign to the rule of three and the "Pythagorean Theorem" the role of a foundation, at least in the context of astronomical mathematics. The *Yuktibhāṣa*, an important treatise of the Kerala school, states that,

> most of mathematical computations are pervaded by this … 'rule of three' and the … 'rule of base height and hypotenuse'…. All arithmetical operations like addition etc. function as adjuncts to the above. (Sarma et al. 2009: 30)

Nīlakaṇṭha, another proponent of the same school, makes a similar statement in the context of astronomical calculations (Śāstrī 1930: 100). A similar exaltation of the rule of three is available also in the *Līlāvatī*'s verse 241 (Sarma 1975: 434). However, these statements should not be taken too seriously as foundational statements. Indeed, the "Pythagorean Theorem" and, to an extent, also the rule of three are themselves subject to justification, and obviously cannot have all Sanskrit mathematical reasoning reduced to them.

Following on this question of foundation, I would like to qualify one of the characterizations of medieval Indian mathematical proofs made by Srinivas

(2015): that proofs proceed from the known or established to that which is to be established. This view might be interpreted as restricting proof to synthetic reasoning. But medieval Indian mathematical proofs (as well as those of many other historical and contemporary mathematical cultures) are often enough heuristic or analytic, in the sense of starting from that which is to be established, and deriving its necessary conditions. A synthetic verification that those conditions are also sufficient is sometimes lacking, and the very distinction between deduction and abduction is not salient in many proofs.

Indeed, when reading a mathematical justification, the sequence of written statements does not necessarily correspond to their inferential order (compare "I came because you called" and "you called, so I came"). The ambiguity of some Sanskrit adverbials and the free word order allowed in Sanskrit verse make it sometimes difficult to decipher the intended logical order – assuming that a clear-cut order was actually intended. Modern reconstructions of proofs in Sanskrit mathematical literature often suppress this ambiguity and impose logical clarity where the sources are ambiguous (note that this is not something special to Sanskrit – translations sometimes prefer – or are forced to – translate an ambiguous term in the original by a univocal term).

The traditional Sanskrit term for proof, *upapatti*, stems from the grammatical root √*pad*, with the prefix *upa*, and has to do with approaching, reaching and occurrence and production, which may suggest a linear advance from established knowledge to new knowledge, or an account of how knowledge is born. But the other term, *yukti*, which has become more popular in Kerala mathematics (Divakaran 2016), has a different semantic field. The root √*yuj* has to do with tying or connecting things together (hence the English "yoke" and Sanskrit *yoga*; *yukta* is one of the terms for the arithmetical adjective "added"). This semantic field does not suggest a directionality of reason, but the integration of different pieces of knowledge.[5] Regardless of whether this etymology reflects a conscious choice or just the unconscious vicissitudes of language, I will show how a principle of integration of knowledge, rather than one of linear progression, manifests itself in the *Kriyākramakarī*.[6]

5 This interpretation is already suggested in a different context by Wujastyk (2003: 25).

6 I note also that the above terms sometimes accompany what we would consider an illustrative example, rather than a proof. Furthermore, these terms do not explicitly accompany all proofs or justifications in the text. Therefore, identifying these terms with contemporary mainstream notions of mathematical proof may be problematic.

2. COMPONENTS OF PROOF

IN THIS SECTION I WILL DESCRIBE the main basic sources of knowledge used in the *Kriyākramakarī*. I did not find sufficiently detailed explicit discussions of the components of mathematical proof in the *Kriyākramakarī* itself, so, as a tentative *ersatz*, I will apply the above quoted epistemological division into perception, tradition and inference. Note, however, that this division is not made salient in the *Kriyākramakarī* or other related mathematical texts, and therefore might not be the authoritative way to organize this material. Moreover, I do not restrict my use of the above categories to their indigenous theoretical meaning. My purpose here is to describe the sources of mathematical knowledge used in the *Kriyākramakarī*, not how its authors would have classified them.

PERCEPTION

An example for an application of perceptual knowledge that stands out most clearly for a reader versed in contemporary mathematical standards is the following. Two parallel bamboo reeds have strings connecting the root of each to the tip of the other.

> When there's equality [in the size of the reeds], the intersection of the strings is in the middle of the space between them; when there's no equality, the intersection of the reeds is near the smaller cane.[7]

But this kind of observational justification referring to non-mathematical entities is not used frequently in the *Kriyākramakarī*.[8]

A more obvious and prevalent perceptual source of knowledge is cut-and-paste geometric arguments.[9] In the history of mathematics, geometric diagrams are most strongly associated to the classical Greek tradition, but they are found in most other mathematical cultures as well. While classical Greek mathematical reasoning depended on a complementary relation between a lettered diagram and a highly formulaic text (Netz 1999), in other cultures diagrams did not depend on letters or on a strongly regimented deductive system.

7 Sarma 1975: 311, verse 2:
साम्ये वेण्वोरन्तरालभूमध्ये सूत्रयोर्युतिः ।
साम्याभावेऽल्पवंशस्य निकटे सूत्रयोर्युतिः ॥

8 The *Yuktibhāṣa* has a striking observational explanation that uses interlocking slanting beams in a roof structure to justify the similarity of some triangles (Sarma et al. 2009: 184 f.).

9 One reviewer suggested that this would not be considered perceptual knowledge by the authors of the *Kriyākramakarī*, due to the absence of actual diagrams in the text. Since the authors do not explicitly analyze proofs using this term, it is hard to decide on this issue.

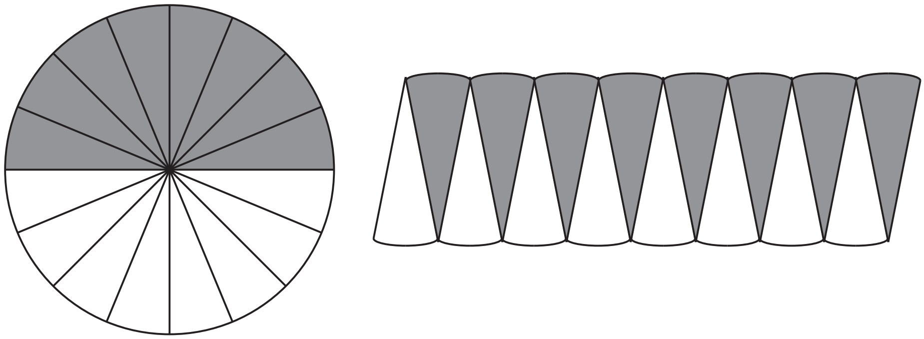

Figure 1: A visual explanation of rearranging a circle as an approximate rectangle (redrawn following Sarma et al. 2009: 264).

A famous anecdote from the Indian tradition is Bhāskara II's diagrammatic proof of the "Pythagorean Theorem" accompanied by a single word: "behold!" But surviving sources do accompany such diagrams with instructions that clarify their meaning. In fact, one is more likely to find a geometric cut-and-paste argument without a diagram than a diagram meant to justify a general claim without the accompaniment of a textual argument. The *Kriyākramakarī* contains quite a few arguments of this kind, especially in the context of quadratic and cubic identities, the summation of progressions, and, of course, geometric area calculations. A less standard diagrammatic argument involves cutting a circle into sectors, and fitting them together to form an approximate rectangle to justify the formula for the area of a circle (Sarma et al. 2009: 264, see Figure 1).

Other sources of knowledge that may arguably be placed under the heading of "perception" (and here we are being anachronistic) include elementary arithmetic rules and combinatorial reasoning. For example, what we would call "the distributivity of multiplication" is introduced in the following verse:

> When a multiplier subtracted by something is multiplied, the multiplicand should be multiplied by that something and subtracted [from the product of the original multiplier and multiplicand]. If something is added, it [the product] should be added.[10]

This could be read as an observation of what happens when one performs a multiplication according to its definition as repeated addition (later this is also explained by cut-and-paste geometry).

10 Sarma 1975: 17, verse 18:
इष्टोनगुणकाभ्यस्ते गुण्य इष्टाहतः स तु।
न्यूनः स्यादिष्टयुक्तेन यदि सोऽभ्यधिको भवेत्॥

In the context of combinatorics, the counting of the number of meters with exactly 8 syllables, of which exactly x are to be long syllables, is explained as follows: successively choose places for the long syllables, and then cancel the repetitions by dividing the total number of arrangements obtained by the number of repetitions of each arrangement. Here again, the number of repetitions is observed, rather than derived. Note that this is true for contemporary combinatorics classes as well.[11]

AUTHORITY

Commentaries can be polemical and argumentative, as was often the case in the Greek, Arabic and Latin cultures. Indian mathematical commentaries tend, however, to be more respectful toward their sources. Even when information has to be complemented or made more precise, they often (but not always!) present it as the fault of the uneducated reader, rather than that of the master, who simply took things for granted. Nevertheless, authority was not followed blindly, and Srinivas (2015) and Divakaran (2016) show that when observation contradicts authority the latter has to retreat or, at least, find excuses (perhaps things have changed since the time of the old masters…).

The *Kriyākramakarī* accepts the authority not only of its source, the *Līlāvatī*, but of many other authors, listed in appendix A. The verses of these authors are sometimes brought simply to state the rules of the *Līlāvatī* in different words, and sometimes to provide additional methods and introduce problems not covered in the *Līlāvatī*. The accumulation of different authorities echoing and complementing each other obviously serves to strengthen the reader's sense of conviction.

INFERENCE (AND OTHER FORMS OF REASONING)

The *Kriyākramakarī* includes one substantial reference to the *anumāna* logic system, relying on it to explain the rule of three: Just as smoke on a mountain indicates a fire, so does smoke in the kitchen; similarly, a given ratio between the price and quantity of some fruit is preserved when a different quantity is to be bought. In both cases there's a common "law" that connects the cause and effect in the two compared situations (see Hayashi 2000: section 3.2, and the translated verses in Appendix E below).

11 If one were to formally derive the counting, one would have to use tedious set theoretic formulations which would obfuscate the argument. But as many mathematicians and philosophers have observed (e.g., Poincaré, Hilbert, Wittgenstein…), following a rigorous formalism still requires the capacity to compare and count symbols by means of observation.

In a broader sense (and again, risking anachronism), one might use the same kind of framework for applying known results in justifications. For example, just as the product of sum and difference equals the difference of squares in one context, so does the equality apply in another. But such inferences are not explicitly presented as applications of *anumāna* in the *Kriyākramakarī*, and are perhaps better treated as naïve deductions that are not theoretically grounded, or, to use Keller's terms, a form of re-interpretation.

Another form of reasoning used in the *Kriyākramakarī* is calculation. This is obvious when it comes to actual calculation with specific numbers in solved examples, but perhaps not so obvious when it comes to calculations applied to undetermined quantities. This is not proper algebra, as it does not use Bhāskara II's algebraic terminology from his *Bījagaṇita*. But calculation algorithms are applied rhetorically to undetermined quantities (e.g., root extraction applied to a term containing an arbitrary square – see the first rule in Appendix B below) in order to derive conclusions (e.g., that it has a rational square root).

Next comes inversion. If a certain procedure leads one from the datum to a result, then reversing the procedure and applying it to the result should retrieve the datum. This is brought up explicitly as a problem solving method (*vyastavidhi*) in the Sanskrit tradition. The *Kriyākramakarī* also uses it to justify some calculation procedures, such as root extraction, which is an inversion of the squaring procedure.

Mathematical induction can also be found in the *Kriyākramakarī*, for example, in the proof that the sum of squares up to n equals

$$\frac{n\,(n+1)\,(2n+1)}{6}.$$

The proof starts by multiplying everything by 6. Then, it considers the last term of the sum, $6n^2$. This term can be interpreted as the number of blocks in the external half-shell (two levels of base and two adjacent walls) of a box of size

$$n \times (n+1) \times (2n+1)$$

Applying the same reasoning to the previous terms fills up the entire box (see Katz 2007: 493–8 for Nilakaṇṭha's version from Śāstrī 1930, which is very similar to that of the *Kriyākramakarī*).[12] While mathematical induction is never stated explicitly as an independent principle, it is used in several geometric proofs of summations.

12 One might object that this is not proper induction because it goes backwards (from n down) rather than forward. But in fact, induction should go backward. The difference is imperceptible in proofs of simple integer formulas, but can lead to confusion in more complex situations.

3. THE ORGANIZATION OF KNOWLEDGE

THIS SECTION WILL CHARACTERIZE the *Kriyākramakarī*'s approach to the organization of knowledge, arguing that it presents an integrative, rather than a hierarchical view of mathematical justification. I will start with the way specific problem types are treated, and then move to the organization of the treatise as a whole.

PERMUTING GIVENS AND UNKNOWNS

Consider, for example, the treatment of arithmetic progressions. In the *Āryabhaṭīya* (fifth century), we are taught how to sum a progression given its initial term, difference and number of terms, and how to derive the number of terms from the other data. The *Līlāvatī* also adds procedures to determine the first term and difference from the other terms (this is easily obtained by inversion). We then get a system of four parameters, any three of which determines the fourth. For the *Kriyākramakarī*, this form of completion is a general principle in organizing the treatment of problems.

For example, in the case of the *Līlāvatī*'s barter problems, one is given the price of a certain amount of one commodity and the price of another amount of another commodity, and is required to compute the exchange rate between the two commodities. The *Kriyākramakarī* adds problems where the price of one commodity is given together with the exchange rate of the two commodities, and one has to calculate the price of the other commodity. A similar approach is taken in the context of the rule of 5, where each parameter, rather than just the yield (*icchāphala*), is separately set as unknown (Sarma 1975: 195 f., 204).

Elsewhere, the *Līlāvatī* discusses the following problem: given two right-angled triangles sharing a given side, find their bases and hypotenuses from

Consider, for example, the following *false* theorem: "All simple graphs without isolated vertices are connected". Here is the inductive "proof." For the case of 1 vertex, the theorem is true by default, and for 2 vertices the only simple graph without isolated vertices is an edge, which is indeed connected. Now suppose the theorem is true for n vertices, so that every graph of n vertices without isolated vertices is connected. Add one more vertex. By hypothesis, it is not isolated, so it is connected by at least one edge to the rest of the connected graph. Therefore, the entire graph connected. Nevertheless, the theorem is wrong for 4 vertices (two disjoint edges have 4 vertices, none of which is isolated). A correct proof should have started with a graph of $n+1$ vertices, none of which is isolated. Then, removing one vertex and its edges, the remaining graph might have an isolated vertex (as in the example of the two disjoint edges above), so the inductive hypothesis could not have be applied, and the proof would fail. The moral of this is that an inductive proof should consider a general $(n+1)^{\text{th}}$ case and cite cases of size n or less to derive the conclusion.

the difference of their bases and the difference of their hypotenuses. The *Kriyākramakarī* complements this problem by considering all combinations of sums and differences of the bases and hypotenuses (Sarma 1975: 423–28).

So far, this is not unique to the *Kriyākramakarī* or to medieval Indian mathematical commentaries. The view of a problem as a set of connected quantities, each of which should be obtained from the others, can be found in many mathematical cultures. What emerges as a feature specific to Kerala mathematics is that it becomes an explicit strategy with a generic title: the so-and-so many questions (*praśna*) or questions and answers (*praśnottara*).

A famous example is Citrabhānu's 21 questions and answers, mentioned above, which survives in the *Kriyākramakarī* (Sarma 1975: 108–26; Hayashi and Kusuba 1998). Here given any two of seven possible combinations of two unknowns (involving sums, products, squares and cubes), the other quantities are to be reconstructed. The *Yuktibhāṣa* too has a section called 10 questions and answers, based on five out of those seven quantities (Sarma et al. 2009: 20–22). Note that in Arabic algebra, the treatment of such questions would be very different: the problems would be reduced to one of the canonical equations in one variable (e.g., $x^2 = bx + c$), and solved using the appropriate procedure. Here, instead of reducing to a basic canonical form, the system of different problems is explored.

This approach applies to astronomical-mathematical problems as well. The *Tantrasaṅgraha* applies the same organization to five spherical-trigonometric quantities, of which any three are given (Ramasubramanian and Sriram 2010: 200–28), and the *Yuktibhāṣa* introduces a set of 15 problems based on six spherical-trigonometric quantities, any two of which are given (Sarma et al. 2009: 533–40). When the *Kriyākramakarī* introduces the problem of the shadow of a sphere on a surface, it is organized as a system of four quantities, such that given any three, the fourth is to be determined (Sarma 1975: 435–37).

We see that an integrative view of problems based on permuting their givens and unknowns becomes an explicit principle for the organization of knowledge in Kerala mathematics. This practice has antecedents elsewhere, and is related to the method of inversion. But the thematization of this practice as a general approach does seem to be specifically endorsed by Kerala mathematical-astronomers. Our next step is to see how this plays out on a larger scale.

RELATING DISTANT PIECES OF KNOWLEDGE

I believe that the view of problems as collections of related quantities, each of which can be determined from the others, is extended in the *Kriyākramakarī* to mathematics as a whole. This means that justifications should not follow a predictable course, building from the more elementary to the more complex, but should instead connect disparate pieces of mathematical knowledge in various directions.

A first example is the treatment of the formula

$$(a + b)(a - b) = a^2 - b^2$$

As one would expect, it is indeed derived from algebraic manipulations (applying the distributive law to the left hand side) as well as from cutting and pasting rectangular diagrams (Sarma 1975: 31–36). What is more surprising is that it is also derived from the fact that the sum of odd integers up to $2n-1$ equals n^2 (Appendix D). This line of reasoning explicitly connects arithmetic progression, the discussion of squares and roots, and the above quadratic identity. It also justifies a rather simple, and already established quadratic identity by means of a more advanced summing of an arithmetic progression, which is only justified later in the book (Sarma 1975: 241).

Another example involves the proof of Heron's formula for the area of a triangle. The proof relies on knowledge about triangles already established in the treatise in order to analyze various proportions in a triangle, eventually combined to produce the formula. But it seems that the author runs into difficulties when attempting to conclude the proof (if my reading is correct – and I am not certain that it is – the author confuses $\sqrt{x^2 + 1}$ with $x + 1$, and then abandons the original line of proof). At that point, the author notes an analogy between the system of proportions that had been established, and the proportions relating arrows, chords and radii in intersecting circles, which goes back to Āryabhaṭa (see Appendix F). This analogy allows to clinch the proof without going through the entire argument. Note that we don't have here an abstract theorem on proportions that is applied to two different situations (circles and triangles). What we have here is an analogy relating two different geometrical problems, one concerning triangles, and the other concerning circles, using one to enhance our understanding of the other.

But the most striking example for my claim is the rich discussion of solutions of quadratic Diophantine variations of the form: "find two squares such that their sum and/or difference together with some given perturbation is a square." Given a solution for such a problem, the most obvious justification would be to plug in the suggested solution, use algebraic identities to rearrange the resulting algebraic expression and verify that the result is a square (we may call this "algebraic synthesis"). While for some rules this is precisely the course taken (rules 0b and 12 as well as the *vargaprakṛti* method in Appendix B), it seems that the *Kriyākramakarī* is intent on exploring as many courses of justification as possible.

Other methods of justification include the following (see Appendix B for details):

- Apply the procedure for root extraction of numbers to the general algebraic term provided by the rule, to show that the resulting sum has a square root (Rule 0a)
- Use cut-and-paste geometry to show that the resulting sum can be rearranged as a square (rule 1)
- In the context of the above, use heuristic reasoning to reconstruct the coefficient that would allow the above procedure to succeed (rule 1)
- Heuristically suggest a form for the solution, find circle segments (Rsines, Rcosines, arrows and radii) that model this solution, and use them to specify solutions that fit the heuristic model (rule 2)
- Heuristically suggest a form for the solution, and use algebraic identities and manipulation to specify solutions that fit the heuristic model (rules 4–6, 10–11, all using different heuristic models; we may call this "algebraic analysis")
- Brahmagupta's *vargaprakṛti* method (a quadratic variant of the *kuṭṭaka* method)
- Deriving difference equations for solutions from specific applications of the last method (rules 7, 8).

We see here an attempt to connect algebraic analysis and synthesis, rectilinear geometry, circle geometry, root extraction, quadratic Diophantine equations, and their recursive solutions by the *vargaprakṛti* method. The distinction between these forms of reasoning is not an anachronism – many of them are indigenously understood as distinct, as reflected in the internal divisions of the *Kriyākramakarī* (see Appendix A). The message, I believe, is one of an underlying unity of mathematics.

The same kind of problem is revisited when discussing the construction of rational right-angled triangles (which, in arithmetic terms, means finding two squares that sum to a square, see Appendix C). Here one uses some algebraic manipulations similar to those already used when the algebraic problem was solved by a geometric model. The *Kriyākramakarī* applies the same treatment, but from an opposite point of view: going from geometry to algebra, rather than vice-versa.

A second solution to the algebraic problem, which is more sophisticated, is justified both by Mādhava's formula for the Rsine and Rcosine of a sum of angles as well as algebraically. Again, we see an attempt to relate as many different points of view to a given family of problems, and no hesitation in applying heavy guns to solve relatively simple problems. At the end of the discussion of the various solutions, they are all united as applications of the rule of three to the same general solution.

We do not have any explicit statements declaring that the authors specifically intend to integrate different aspects of mathematics. One may therefore claim that the purpose is simply to explore many different solutions for each problem (with no intention of integrating mathematical knowledge), or simply to demonstrate virtuosity.

The latter claim seems unlikely, as the *Kriyākramakarī* often cites and credits others, without ever explicitly claiming anything to be an original invention. The former interpretation, however, is not as easy to rule out. To argue against it, recall that some of the justification and solutions presented are highly inefficient and contrived. If the authors simply wanted to propose different kinds of solutions, they could have achieved this goal by simpler means. Moreover, the "directionality" of knowledge (going from the simple or established to the advanced or not yet known) is not preserved. As this might raise more suspicion than conviction, it's unlikely to assume that the authors' goal was simply variety of justifications, and even if it was, the effect generated is that of an integration of mathematical knowledge, where distant components end up relating to each other. I therefore conclude that the authors sought to present mathematics as a unified whole, rather than as a system of separate problems, each having several different solutions.

I do not claim that integrating distant pieces of knowledge is the only purpose of proofs in the *Kriyākramakarī*. It is clear that the proofs were meant to convince and explain, and it is clear that the diversity of methods presented in the treatise teaches the reader different approaches to problems. Such approach can be attested elsewhere, for example in the context of Pṛthūdaka's nineth-century mathematical commentary *Vāsanābhāṣya*, which Keller (forthcoming) characterizes as trying,

> to make sense of the variety of possible understandings of the text, taken in itself. Such an attitude has often been noted in other scholarly disciplines of South Asia as well.

But in the case of *Kriyākramakarī*, the overall picture suggests an attempt to relate distant aspects of mathematics in the presentation of mathematical knowledge.

4. CONCLUSION

In this paper I studied the system of justifications in the *Kriyākramakarī*. The variety of proof methods and the organization of knowledge suggest that the authors are interested in presenting an integrative view of mathematical knowledge, which emphasizes the links between different domains of mathematics.

Within this context, the statements that appear to "reduce" mathematics to the rule of three or the "Pythagorean theorem" turn out to attempt not some naïve version of a foundational approach, but rather an integration of mathematics. This integration is achieved not only by framing many forms of mathematical reasoning in terms of the two tools above. It is achieved by exploring how we can move back and forth between various parameters that exchange the roles of data and solution in given problems, and by linking disparate areas of mathematics through mutual justifications.

We must note, however, that the findings presented here are highly restricted. In order to present a more general picture concerning the organization of knowledge in Kerala mathematics, we should compare the *Kriyākramakarī* to other Kerala based commentaries. The *Yuktibhāṣa*, for instance, is not interested in ingenious justifications of simple mathematics, and glosses over well established mathematics rather quickly. It is clear that its main interest is to provide justifications for the most advanced pieces of mathematical knowledge applicable to astronomy. Nevertheless, even the *Yuktibhāṣa* sometimes presents several justifications for a single result.

How does this picture relate to other proponents of the Kerala school (e.g., Nīlakaṇṭha)? What about practitioners of other sciences in the same social system or elsewhere? Is a concern with integrating knowledge reflected in indigenous philosophical-logical debates? And how does this relate to vernacular or practical mathematics? All these questions will require further research.

5. APPENDIX A: SUMMARY OF THE CONTENT OF THE *LĪLĀVATĪ* AND THE ADDITIONS OF THE *KRIYĀKRAMAKARĪ* (KKK)

In the following table, the verse numbering in the second column follows Sarma 1975.

In the third column, when the name of a treatise is not specified, the treatise is unknown. The entries are derived from Sarma's edition, corrections of one of the reviewers and additional observations in Vrinda 2017.

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Invocations; units of measurement (money, length, area, volume)	1–8 1–9	Śrīdhara's *Pāṭīgaṇita*, Śrīpati, Ārybhaṭa's *Ārybhaṭīya*	Additional measurement units including time units
Names of decimal multiples	8 10–11	*Pāṭīgaṇita*	
Decimal addition and subtraction	9–10 12–13		
Multiplication by place value, by breaking a factor into a sum or a difference, and by breaking a factor into a product	10–19 14–17	*Pāṭīgaṇita*	Multiplication of a sum or difference by another sum or difference, modeled and explained by a cut-and-paste geometry.
Division by place value and reducing the dividend and divisor by a common factor	19–22 18	Govindasvāmin, *Pāṭīgaṇita*	Greatest common divisor algorithm
Squaring by place value; the following combinations yield squares: $a^2 + b^2 + 2ab$; $(a-b)^2 + 4ab$; $(a+b)(a-b) + b^2$	22–39 19–21	Govindasvāmin, *Pāṭīgaṇita*, Śrīdhara's *Triśatikā*, Parameśvara's *Līlāvatī* commentary	Cut-and-paste and algebraic justification of the square formulas. Squaring by parts: $(a+b)^2 = a^2 + b^2 + 2ab$; squaring by division: $x^2 = \left\lfloor \frac{x}{y} \right\rfloor^2 y^2 + 2\left\lfloor \frac{x}{y} \right\rfloor y \left\{ \frac{x}{y} \right\} + \left\{ \frac{x}{y} \right\}^2$, where $\left\lfloor \frac{x}{y} \right\rfloor$ and $\left\{ \frac{x}{y} \right\}$ are the quotient and residue respectively. Arithmetic progression sum formula applied to $1 + 3 + \ldots + 2n - 1$ to show that it yields n^2.

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Extracting square roots by place value	39–44 22–23	*Pāṭīgaṇita*, Govindasvāmin, Parameśvara's *Līlāvatī* commentary	Extraction of non-integer roots (See Appendix B, rule 0a). Justification of procedure by inversion (*parāvṛtti*) of squaring. Derivation of $(a+b)(a-b) = a^2 - b^2$ from the above progression of odd numbers (Appendix D).
Cubing by place value; the following yield cubes: $a^3 + b^3 + 3ab(a+b)$, $a(a+b)(a-b) + ab^2$	44–55 24–27	Govindasvāmin, Brahmagupta's *Brāhmasphuṭasiddhānta*, *Pāṭīgaṇita*, Parameśvara's *Līlāvatī* commentary, *Āryabhaṭīya*	Geometric 3D modeling and cut-and-paste justification of $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$ $= a^3 + b^3 + 3ab^2 + 3a^2b$ $a^3 = a(a+b)(a-b) + ab^2$
Extracting cubic roots by place value	55–58 28–29	*Pāṭīgaṇita*, Govindasvāmin, *Brāhmasphuṭasiddhānta*, Parameśvara's *Līlāvatī* commentary	Root procedure explained as inversion of the cubing procedure
Fractional combinations: plain fractions (*bhāga*) with standard common denominator addition; fractions of fractions (*prabhāga*) combined multiplicatively, and attached fractions (*bhāgānubandha*), namely additive combinations of number and fraction and the combination of $\frac{a}{b}$ with $\frac{c}{d}$ as $\frac{a}{b} \pm \frac{a}{b} \bullet \frac{c}{d}$.[12]	59–74 30–36	Parameśvara's *Līlāvatī* commentary, Govindasvāmin's *Gaṇitamukha*, *Pāṭīgaṇita*	Finding the least common multiple. Analogy between fractions of fractions and the rule of three. Generalization of the first kind of *bhāgānubandha* from number and fraction to standard sum of fractions

[12] The apparent redundancy within this division and with respect to the next item is most likely due to the degeneration of a previous more complex system of fractions (cf. Heroor 2014: 5–33).

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Arithmetic operations on fractions (*bhinna*)	75–90 37–44	Parameśvara's *Līlāvatī* commentary, Govindasvāmin, *Pāṭīgaṇita*, Śrīdhara, *Brāhmasphuṭasiddhānta*, *Āryabhaṭīya*, Ācārya (unidentified)	Elementary justification of procedures (especially division), a place value (not necessarily decimal) approach to squaring and root extraction of fractions.
Operations with zero. Division by zero results in "zero divided" (*kha-hara*); multiplying and dividing by zero cancel out	91–94 45–47	*Āryabhaṭīya*	Discussion of the cancellation of multiplication and division by zero in terms of inverse operations. Applications to astronomy, where procedures that involve multiplying and dividing by trigonometric functions may degenerate into multiplying and dividing by zero in the case of 90° or 0°.
Inversion (*vyastavidhi*): recovering an unknown by inverting the operations applied to it	95–99 48–50	*Pāṭīgaṇita*, Govindasvāmin	Additional examples, including astronomical examples. Relation between successive multiplication and division and the rule of three.
False position (*iṣṭa-karma*)	100–107 51–55	*Triśatikā*, *Pāṭīgaṇita*, Govindasvāmin	Analogy to inversion and the rule of three.
Reconstruction of two unknowns from their sum and difference and from their difference of squares and difference	108–126 56–59	Citrabhānu	Citrabhānu's 21 questions: how to reconstruct two unknowns from any pair in the following list: their sum, difference, product, sum of squares, difference of squares, sum of cubes, difference of cubes. The commentary includes examples and algebraic and geometrical proofs (see Hayashi and Kusuba 1998).
Quadratic Diophantine problems (*vargakarma*)	127–167 60–63	Parameśvara's *Līlāvatī* commentary, Udayadivākara, Jayadeva, *Brāhmasphuṭasiddhānta*	Many additional methods of solution as well as a general treatment of *vargaprakṛti* ($x^2 = Ny^2 \pm c$), deriving rational, but not necessarily integer solutions (see Shukla 1954; Appendix B). The commentary skips or misses verse 64 that introduces the term *bījagaṇita* (algebra).

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Quadratic equations (*guṇakarma*) solved by completing to square	168–177 65–72		A combination of geometric cut-and-paste arguments and algebraic reasoning to justify the solution
Rule of three (*trairāśika*): a given measure (*pramāṇa*) yields a known result (*phala*), what would be the desired yield (*icchāphala*) of a known desired measure (*icchā*)? In standard rule of 3: *pramāṇa* : *phala* :: *icchā* : *icchāphala*. In inverse rule of 3: *pramāṇa* : (1/*phala*) :: *icchā* : (1/*icchāphala*). Rule of five, etc.; Barter.	178–209 73–89	Govindasvāmin, *Āryabhaṭīya*, *Pāṭīgaṇita*, *Brāhmasphuṭasiddhānta*, *Mahābhāskarīya*, Govindasvāmin's commentary on the latter and his *Govindakṛti*, Śrīpati, Bhāskara I's *Āryabhaṭīyabhāṣya*	Logical interpretation of rule of three in terms of analogies of cause and effect (see Hayashi 2000; Appendix E). Explaining the rule in terms of yield per unit measure and other ratios. The proportion *pramāṇa* : *icchā* :: *phala* : *icchāphala*. Simplifying problems with fractions by rescaling and by moving denominators from one side to numerators on the other. Additional examples, including astronomical ones. Reduction of rule of five to rule of three by multiplying the terms on the *pramāṇa* and *icchā* sides and by rescaling some of the terms in the problem to 1.
Miscellaneous problems based on proportions and rule of three: interest, partnership, filling a well by pipes of different capacity, buying several products for a given sum of money in a given proportion, exchange of gems,[13] mixing gold of different qualities and quantities	210–237 90–111	*Brāhmasphuṭasiddhānta*, *Pāṭīgaṇita*, Śrīpati, Śrīdhara, *Āryabhaṭīya*, Sūryadevayajvan's commentary on the former	Explaining the proportions involved in the problems and expressing solutions in terms of the rule of three or elementary arithmetical reasoning. Several additional examples. After the *Līlāvatī*'s gems problem, the following problem is added: we are given the total property of all people in a group less that of each person separately. How much property does each person have? The commentary discusses also the case where we are given the property of all less that of each pair.

[13]Each person has a distinct kind of gems of a given number and unknown value. They all give a fixed number of their gems to all the others. After the exchange, each person has the same value in gems. How much is each kind of gem worth?

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Sums of $1, 2, \ldots, n$; sum of these sums; sums of their squares and cubes	238–246 112–115	*Brāhmasphuṭasiddhānta*, *Pāṭīgaṇita*, *Āryabhaṭīya*, Bhāskara I's *Āryabhaṭīyabhāṣya*	Additional examples, proof of first sum by the arithmetic progression formula. Inductive geometric 2D and 3D cut-and-paste proofs of the other formulas (Sarasvati Amma 1979). Generalization to further sums of sums of the first progression, but without proof as "the rationale is not easy to understand".[14]
Arithmetic progressions: calculation of sum, last term, middle term, first term, difference and number of terms from each other	246–256 116–124	*Triśatikā*, *Āryabhaṭīya*, Bhāskara I's *Āryabhaṭīyabhāṣya*, *Pāṭīgaṇita*, Parameśvara's *Līlāvatī* commentary	Additional examples; summing an arithmetic progression starting from its m^{th} term, the notion of a "middle term" when there is an even number of terms. Justification of procedures by symmetry arguments, inversion, and, for the quadratic procedure for retrieving the number of terms, a geometric cut-and-paste argument.
Geometric progressions: calculation of last term by a sequence of multiplications and squarings; calculation of sum	256–266 125–127		Additional examples involving *kaṭapayādi* notation. Simplification of the formula for fractional quotients by moving denominators around. Justification of sum formula by arithmetical reasoning (telescoping progression). Two manuscripts add the following problem: everyday a quantity is multiplied by a given variable factor and decreased by a fixed unknown quantity. After a given number days nothing is left. What is the fixed decrease?

[14] Sarma 1975: 243, verse 24: तद्युक्तिः सुगमा न स्यादिति नेह प्रपञ्च्यते।

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Combinatorics: number of meters consisting of 4 legs of 8 short or long syllables, with restrictions on equality or difference of the legs; number of meters with a fixed number of long syllables.	267–276 128–134	Kedara's *Vṛttaratnākara*,	Discussion of examples of similar combinations from other contexts. Justification of second rule by counting free places and dividing by number of repetitions. Note of symmetry between number of long and short syllables.
Right angled triangles: calculating a side given the other sides; useful quadratic identities; approximation of irrational roots by rescaling (multiply by x^2, extract root, divide by x)	277–286 135–140	*Āryabhaṭīya*, Śrīdhara	Cut-and-paste proof of the "Pythagorean" theorem. The root approximation algorithm (see Appendix A, rule 0a). Algebraic justification of rescaling for root approximation.
Sides of rational right angle triangles: $\frac{2nA}{n^2-1}, A, \frac{(n^2+1)A}{n^2-1}$ $\frac{2nA}{n^2+1}, \frac{(n^2-1)A}{n^2+1}, A$ $A, \frac{A^2-n^2}{2n}, \frac{A^2+n^2}{2n}$. Construction of right angled triangles with integer sides.	286–297 141–148	Mādhava	Justifications using a geometric model (chords in a circle), algebra and the rules for sines and cosines of sums of angles. Unification of all rules under a single one by means of the rule of three (see Appendix C).

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Miscellaneous problems with right angled triangles: finding the sides of the triangle from: a side and the sum of hypotenuse and other side / a side and the difference of hypotenuse and other side / a side and part of other side, given that their sum equals the hypotenuse and remaining part of the other side ("jumping monkeys") / the hypotenuse and sum of sides; finding the intersection point of the hypotenuses of two opposing right angle triangles on the same base	297–312 149–162	Govindasvāmin, *Āryabhaṭīya*, Parameśvara's *Līlāvatī* commentary	Justification of the jumping monkeys solution by algebraic manipulations and the rule of three. Algebraic justification of solution of next problem. Justification of last solution by symmetry and rule of three considerations.
General triangles: triangle inequality; calculation of height and the base segments it defines; Brahamagupta's formula as approximate area for a general quadrangle and as exact area for a triangle (equal to Heron's formula)	313–326 163–169	*Brāhmasphuṭasiddhānta*,	Algebraic justification of the height and base segment calculation. Examples and proof of the area formula for triangles (see Appendix F). The commentary misses or skips verses 170–171 that discuss the insufficiency of the sides to determine the area of a quadrilateral.

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Quadrilaterals: rhombus, rectangle, trapezium; calculating diagonals, heights and base segments from each other; area of a general quadrangle; Brahamagupta's and Bhāskara's rules for diagonals; in some versions: Brahamagupta's area formula for quadrangle circumscribed in circle	326–366 172–190	Parameśvara's *Līlāvatī* commentary, Mādhava	Some geometrical justifications, including detailed proofs of Brahamagupta's formulas. Review of Mādhava's rules for the Rsine and Rcosine of a sum of angles and calculating sine tables as in the *Yuktibhāṣa* (Sarma et al. 2009).
Needle shapes (*śūci*): calculating various heights, projections and diagonals emerging from setting two triangles on the same base	367–375 191–198		Some justifications based on previous geometric discussions and the rule of three
Circles and spheres: circumference, area, surface area and volume	376–399 199–203	*Āryabhaṭīya*, Mādhava, *Triśatikā*, Śrīpati, *Brāhmasphuṭasiddhānta*	Various approximations of π, detailed justifications of Mādhava's approximation procedures for arcsines, areas and volumes, very close to the text of the *Yuktibhāṣa* (Sarma et al. 2009). Śaṅkara's commentary ends at verse 199, and Nārāyaṇa picks it up.
Circles: calculation of arrow, chord and diameter from the other two; sides of inscribed regular polygons with 3–9 sides; approximation of chord from arc	403–409 204–213		

Līlāvatī subjects	KKK pages; *Līlāvatī* verses	Additional known sources quoted	Notable additions in KKK
Applications: volume calculation by averaging the areas of sections; volume of pyramids, cones and boxes; wages for cutting different materials based on size, number of sections and the material; height of heaps of different grains relative to the base	410–422 214–231		On pages 399–402 there are verses that measure some prism and a tetrahedron.
Shadows: calculation of the base of a triangle from the difference of the sides and the difference of the base segments defined by the height; calculating shadows of poles given a light source and the position of a light source from shadows; a statement that all computations using division and multiplication are based on the rule of three	423–437 232–241	Parameśvara's *Līlāvatī* commentary, "कश्चिद्गणितयुक्तिविदग्रेसर"	Extension of the first problem to all combinations of sums or differences of base segments and sides. Given a light source projecting the shadow of a sphere on a surface, calculate the height of the light source, the diameter of the shadow, the diameter of the sphere and the height of the light source above the center of the sphere from the other three.
Pulverization (*kuṭṭaka*): integer solutions x, y for equations of the form $ax + b = cy$ (cf. Keller 2006: 128–66).	438–458 242–260	Govindasvāmin, *Mahābhāskarīya*, Parameśvara, *Āryabhaṭīya*, Bhāskara I's *Āryabhaṭīyabhāṣya*	Additional astronomical examples. A discussion of *sāgrakuṭṭaka*.
Combinatorics: complete and partial permutations of a set of digits; combinations of a set of digits with repetitions; sums of these permutations and combinations; partitions	459–466 261–269		The verses 270–272 on partitions and the closing verse are missing or skipped.

6. APPENDIX B

DISCLAIMER: the reconstructions here and in the following appendices misrepresent several aspects of the original text. First, they use modern notation, so they do not reflect the internal logic of the original terms and what these terms do and do not render salient. Second, these reconstructions do not represent the rhetorical structure of the original argument, in terms of omission, emphasis and patterns of expression. Finally, they are not even completely true to the logical structure of the original, as the reconstructions turn elliptic verses into a linear chain of arguments, imposing on the text my interpretations concerning the logical order of statements and some gaps (but when I fill in the larger gaps, I do note it explicitly). I allow myself all this because the purpose of this paper is to explore the organization of relations between different aspects of mathematical knowledge, and not the detailed structures of specific proofs. But the reader should be careful when drawing conclusions on a fine grained scale from this bird's-view reconstruction.

I use Sarma's (1975) numbering of the rules. The first, unnumbered rules are marked as rules 0a and 0b. Rules 0–1 are cited from the *Līlāvatī*. A reviewer noted that rules 2–11 (with the exception of two items under rule 3) are quoted from Parameśvara's commentary on the *Līlāvatī*, and that rule 12 is due to the *Brāhmasphuṭasiddhānta*.

RULE 0A (128–138)

Take any number x. Let $F = 4x - \frac{1}{2x}$ and $S = \frac{F^2}{2} + 1$. Then both $S^2 - F^2 - 1$ and $S^2 + F^2 - 1$ have roots.

Proof

1. The definition of S implies that $F^2 = 2S-2$. Therefore $S^2-F^2-1 = S^2-2S+1$. Its root is $S - 1$.
2. The proof that S^2+F^2-1 has a root depends on the algorithm for extracting a root. This algorithm can be described as follows: in order to calculate the root of $a^2 + b$

   a. Define the divisor to be $2a$.
   b. Define the dividend to be b.
   c. Divide the dividend by the divisor to obtain a possibly approximate value q.
   d. Update the divisor to $2a + q$.
   e. Update the dividend to $\left(2a + q\right) q - b$.
   f. Update the divisor to $2a + 2q$.

g. Half the divisor is the approximate root.
h. Repeat steps c-f; in even iterations of the procedure, q's are subtracted rather than added, and the subtraction in step e is reversed.
i. If the difference in step e is zero, the exact root is obtained.

3. To calculate the root of $S^2 + F^2 - 1$, set the divisor to $2S = F^2 + 2$ and the dividend to $F^2 - 1$.
4. The approximate quotient is $q = 1$.
5. Updating the divisor and dividend according to step c–e, we obtain $F^2 + 4$ and 4 respectively.
6. The next quotient is $\frac{4}{F^2+4} = \frac{4}{16x^2+1/4x^2}$, which is approximately $q = \frac{1}{4x^2}$.
7. Updating the divisor and dividend according to step h we get $16x^2 - \frac{1}{4x^2}$ and 0 respectively.
8. The exact root is therefore $8x^2 - \frac{1}{8x^2}$

RULE 0B (128–138)

Take any number x. Let $F = x + \frac{1}{2x}$ and $S = 1$. Then both $F^2 - S^2 - 1$ and $F^2 + S^2 - 1$ have roots.

Proof (very succinct, so this is a tentative reconstruction)

1. $\left(x + \frac{1}{2x}\right)^2 + 1^2 - 1 = \frac{1}{4x^2}\left(\left(4x^2+1\right)^2 + 4x^2 - 4x^2\right) = \frac{1}{4x^2}\left(4x^2+1\right)^2$, which is a square.
2. $\left(x + \frac{1}{2x}\right)^2 - \left(x - \frac{1}{2x}\right)^2 = 2$, so $\left(x + \frac{1}{2x}\right)^2 - 1^2 - 1 = \left(x + \frac{1}{2x}\right)^2 - 2 = \left(x - \frac{1}{2x}\right)^2$, which is a square.

RULE 1 (138–144)

Take any number x. Let $F = 8x^4 + 1$ and $S = 8x^3$. Then both $F^2 - S^2 - 1$ and $F^2 + S^2 - 1$ have roots.

Proof

1. $(8x^3)^2 = 8 \times 8 \times x^6 = 8x^4 \times 8x^2$.
2. Therefore, S^2 equals a rectangle of length $8x^4$ and width $8x^2$. Break it into two strips of width $4x^2$ each.
3. Arrange the strips around a square with side $8x^4$. To complete the square, a corner of area $(4x^2)^2$ is missing.
4. The square of F consists of the square of side $8x^4$ added to $2 \times 8 \times x^4 + 1$.

The former term equals the original square from step 3, and the latter term equals the missing corner +1.

5. Therefore, $F^2 + S^2$ exceeds the full square by 1.
6. For subtraction, take the strips away from the initial square, and proceed similarly.
7. Consider the coefficient (*guṇaka*) g of x^3 in S and the coefficient k of x^4 in F. We need that $g = k$ and $\frac{(g/2)^2}{2} = k$, so that the equalities in step 4 will work.
8. So we need to find g such that $\frac{(g/2)^2}{2} = g$. If we choose $g = 5$ or 6, the left hand side is a fraction. If we choose $g = 12$, then we get $\frac{(g/2)^2}{2} = 18$. Setting $g = 8$ fits.

Note: the text includes three different summary verses that explain and prove the procedure in similar ways.

RULE 2 (144–147)

Take any two numbers x, y. Let $F = x$ and $S = \frac{x^2-y^2}{2y}$. Then $F^2 + S^2$ has a root.

Proof

1. This proof depends on the formula $a^2 = (a-b)^2 + 2b(a-b) + b^2$.
2. Note that if $(a-b)^2 + 2b(a-b)$ has a root, then adding b^2 will yield a sum of squares with a root. Set $b^2 = F^2 = x^2$, and find a such that $(a-b)^2 + 2b(a-b)$ is a square, playing the role of S^2.
3. To find such numbers, consider a circle with a right angled triangle lying on the radius. The sides of the triangle are B (*bhuja*) and K (*koṭi*), the diagonal (*karṇa*, *śruti*) is D, and the arrows (*śara*) that extend from the sides to the circumference are Z_B and Z_K respectively (see Fig. 2).
4. In a circle, $B^2 = 2KZ_K + {Z_K}^2$ (this follows from the proportional intersection of chords in a circle, which is stated later in the *Līlāvatī*). Since $Z_K = D - K$, this fits the situation in step 2, with $a = D$ and $b = K$.
5. Note that $B^2 + K^2 = D^2$, as promised in step 2.
6. To show that $S = B$ is a rational number (given rational $K$and Z_B), note that analogously to the situation in step 4, $K^2 = 2BZ_B + {Z_B}^2$, and therefore $B = \frac{K^2-{Z_B}^2}{2Z_B}$. Setting $y = Z_B$ completes the construction.

To explicitly complete the argument, note that if we start with any rational $x = K$

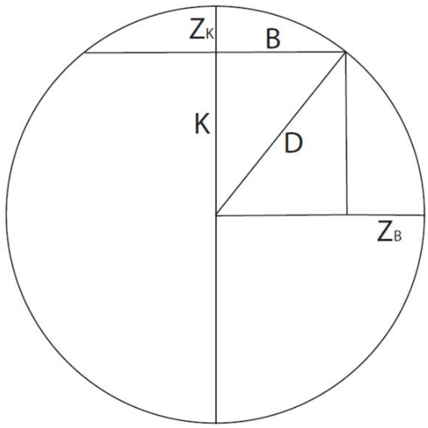

Figure 2: Chords in a circle represent the solution of a Diophantine equation.

and $y = Z_B$, we obtain rational B and D (by steps 6 and 4), so the proof is complete.

The text also states that the sum of squares of x and of $\frac{x^2+y}{4} - 1$ together with y is a square, but this is accompanied only by examples, and no explanation or proof.

RULE 3 (147–148)

Five rules are given with examples but without justification:

- Consider $\frac{(x^2-1-y)}{2}$ and x. The sum of their squares less y is a square.
- Consider $\frac{(x^2-8)}{4}$ and x. The sum of their squares less 4 is a square.
- Consider $\frac{(x^2-9-4)}{6}$ and x. The sum of their squares less 4 is a square.
- Consider $\frac{(x^2+y)}{4} - 1$ and x. The sum of their squares less y is a square (this appeared under rule 2 as well).
- Consider $\frac{(x^2-y^2-1)}{2y}$ and x. The sum of their squares less 1 is a square. This rule is brought as an alternative for cases where the subtraction of 1 in two of the rules above is not possible, but it's not clear to which rules this refers. The list of rules may have been garbled.

RULE 4 (149)

Take any two numbers x, y. Let $F = \frac{x^2+y^2}{2y}$ and $S = x$. Then $F^2 - S^2$ has a root.

Proof

1. $(A + B)(A - B) = A^2 - B^2$.
2. $(A + B) + (A - B) = 2A$.
3. Multiplying the latter by $A-B$ yields $(A + B)(A - B) + (A - B)^2 = 2A(A - B)$.
4. Dividing by $2(A - B)$ yields $\frac{(A+B)(A-B)+(A-B)^2}{2(A-B)} = A$.

The proof ends there with no further details. I suggest the following heuristic reconstruction:

1. We are looking for two numbers whose difference of squares is a square, say $A^2 - x^2 = B^2$.
2. Since in this case $x = (A + B)(A - B) = A^2 - B^2$, Substituting this into the previous formula, we get $\frac{x^2+(A-B)^2}{2(A-B)} = A$.
3. Letting $y = A - B$ we obtain $\frac{x^2+y^2}{2y} = A$.

Alternatively, the identity in step 4 may refer to the geometric model from rule 2.

RULE 5 (149–150)

Take any two numbers x, y. Let $F = \frac{x^2+1+y}{2}$ and $S = x$. Then $F^2 - S^2 - y$ has a root.

Proof

1. $2A - 1 = A^2 - (A - 1)^2$, so $A = \frac{A^2-(A-1)^2+1}{2}$.
2. If $2A - 1 = x^2 + y$, we get $A = \frac{x^2+y+1}{2}$ and $A^2 - x^2 - y = (A - 1)^2$, which is a square.

RULE 6 (150)

Take any two numbers x, y. Let $F = \frac{x^2-y}{4} + 1$ and $S = x$. Then $F^2 - S^2 + y$ has a root.

Proof

1. $4(A - 1) = A^2 - (A - 2)^2$.

2. If $4(A-1) = x^2 - y$, we get $A = \frac{x^2-y}{4} + 1$ and $A^2 - x^2 + y = (A-2)^2$, which is a square.

RULE 7 (151)

Set $y_1 = 2, y_2 = 12$. Set $y_3 = 6y_2 - y_1$, and so on. Then $2{y_n}^2 + 1$ is a square.

This rule is accompanied by examples, but no justification. According to the *vargaprakṛti* method (see the end of this appendix), once one has guessed the solution $(x_1, y_1) = (3, 2)$ to the equation $x^2 = 2y^2 + 1$, any other solution (x_n, y_n) yields a further solution $(x_{n+1}, y_{n+1}) = (3x_n + 4y_n, 2x_n + 3y_n)$. From these two difference equations one can derive the equation $y_{n+1} = 6y_n - y_{n-1}$. I will not try to speculate on the precise derivation of this equation.

RULE 8 (151–152)

Set $y'_1 = 1, y'_2 = 5$. Set $y'_n = 5y'_{n-1} + 4(y'_{n-2} + \ldots + y'_1)$. Then $2{y'_n}^2 - 1$ is a square.

This rule is accompanied by examples, but no justification. Again, according to the *vargaprakṛti* method, once we guess the initial solution $(x_1, y_1) = (1, 1)$ to the equation $x'^2 = 2y'^2 + 1$, any solution (x_n, y_n) of the previous problem yields a solution of the new problem of the form: $(x'_n, y'_n) = (x_n + 2y_n, x_n + y_n)$. Combining this with the difference equations from rule 8 allows to derive the required difference equation. Again, I will not try to speculate on the precise derivation of this equation.

RULE 9 (152–153)

Take any two numbers x, y. Let $F = \frac{x^2-y^2}{2}$ and $S = \frac{x^2-y^2}{2} + y^2 \mp 1 = x^2 - \frac{x^2-y^2}{2} \mp 1$. Then both $S + F \pm 1$ and $S - F \pm 1$ have roots.

This rule is accompanied by examples, but no justification.

RULE 10 (153)

Take any two numbers x, y. Let $F = \frac{x^2 \pm 1}{y}$ and $S = y$. Then $FS \mp 1$ is a square.

Proof

1. The product has to be a square ± 1.
2. $\frac{a}{b} b = a$. Setting $a = x^2 \pm 1$ and $b = y$ concludes the proof.

RULE 11 (154)

Take any three numbers x, y, z. Let $F = \frac{1}{2}\left(\frac{x^2-y^2}{z} + z\right)$ and $S = \frac{1}{2}\left(\frac{x^2-y^2}{z} - z\right)$. Then $F^2 - S^2 = x^2 - y^2$.

Proof

1. $(A+B)(A-B) = A^2 - B^2$, so $A + B = \frac{A^2-B^2}{A-B}$ and vice versa.
2. $A = \frac{(A+B)+(A-B)}{2}$, $B = \frac{(A+B)-(A-B)}{2}$.
3. Setting $A + B = \frac{x^2-y^2}{z}$, $A - B = z$ and applying the previous formula yields $A = F$ and $B = S$. By step 1, the difference of their squares is $(A+B)(A-B) = x^2 - y^2$.

RULE 12 (154–158, CITED FROM PARAMEŚVARA)

Take any two numbers x, y. Consider $A_1 = x^2+y^2$, $A_2 = x^2-y^2$ and $B = \frac{A_1+A_2}{\left((A_1-A_2)/2\right)^2}$. Let $F = A_1B$ and $S = A_2B$. Then $F + S$, $F - S$ and $FS + 1$ are all squares.

Proof

1. In the above setting, $B = \frac{2x^2}{y^4}$.
2. To obtain $F + S$ and $F - S$, we multiply B by $A_1 + A_2$ and $A_1 - A_2$, that is by $2x^2$ and $2y^2$.
3. The quotient of squares is a square, and the same goes for the quotient of a half square and a double square. The product of squares is a square, and the same goes for a product of double squares.
4. In step 2 we have a product of two double squares divided by a square, which is a square. The roots of $F + S$ and $F - S$ are $\frac{2x^2}{y^2}$ and $\frac{2x}{y}$ respectively.
5. By adding the above sum and difference, we can derive $F = \frac{2x^4}{y^4} + \frac{2x^2}{y^2}$ and $S = \frac{2x^4}{y^4} - \frac{2x^2}{y^2}$.
6. Extracting $2x^2$ from each term, we get $FS = \left(\frac{x^2}{y^4} + \frac{1}{y^2}\right)\left(\frac{x^2}{y^4} - \frac{1}{y^2}\right)(2x^2)^2 = \left(\frac{2x^4}{y^4}\right)^2 - 2\frac{2x^4}{y^4}$.
7. Since this is a square less twice its root, adding 1 completes this difference to a square. Its root is $\frac{2x^4}{y^4} - 1$.

VARGAPRAKṚTI (158–167)

Following the rules presented above, the text presents the *Vargaprakṛti* method dealing with problems of the form $x^2 = Ny^2 \pm c$. The solution follows Udayadivākara's commentary on Jayadeva's elaboration of Brahmagupta's work (Shukla 1954). Only the first seven of Jayadeva's twenty verses are included. These are enough to find rational solutions for $c = 1$, but does not provide the full *cakravāla* method for finding integer solution of the general equation. The method presented is the following:

- If $x_1{}^2 = Ny_1{}^2 + c_1$ and $x_2{}^2 = Ny_2{}^2 + c_2$, then $\left(x_1x_2 + \mathrm{Ny}_1y_2\right)^2 = N\left(x_1y_2 + x_2y_1\right)^2 + c_1c_2$.
- Taking arbitrary y_1, one can choose x_1 such that its square approximates $Ny_1{}^2$. The difference is set as c_1.
- Composing this solution with itself as above, we get a solution for $x^2 = Ny^2 + c_1{}^2$.
- Dividing x and y by c_1 solves the equation $x^2 = Ny^2 + 1$.

The discussion of the method includes a proof of the composition rule:

1. $\left(x_1y_2 + x_2y_1\right)^2 = x_1{}^2y_2{}^2 + x_2{}^2y_1{}^2 + x_1y_2x_2y_1 + x_2y_1x_1y_2$. Each of the four summands is a product of four terms. This is multiplied by N and added to c_1c_2.
2. $\left(x_1x_2 + \mathrm{Ny}_1y_2\right)^2 = x_1{}^2x_2{}^2 + N^2y_1{}^2y_2{}^2 + x_1x_2Ny_1y_2 + Ny_1y_2x_1x_2$. We again have four terms, two of which are products of five terms.
3. The cross product of five terms in the squares from steps 1 and 2 (after the former is multiplied by N) are identical. We are left with having to show that $N\left(x_1{}^2y_2{}^2 + x_2{}^2y_1{}^2\right) + c_1c_2$ equals $x_1{}^2x_2{}^2 + N^2y_1{}^2y_2{}^2$.
4. Note that $y_i{}^2 = \frac{x_i{}^2 - c_i}{N}$. The proof is not detailed any further, but one can see that we get on the left hand side $x_1{}^2\left(x_2{}^2 - c_2\right) + x_2{}^2\left(x_1{}^2 - c_1\right) + c_1c_2$ and on the right hand side $x_1{}^2x_2{}^2 + \left(x_1{}^2 - c_1\right)\left(x_2{}^2 - c_2\right)$. The equality is now easy to justify.

7. APPENDIX C

THE FOLLOWING IS THE JUSTIFICATION that for given rational K and n, the three numbers K, $\frac{2nK}{n^2-1}$ and $\frac{(n^2+1)K}{n^2-1}$ form a right angled triangle with rational (*akaraṇīgata*) sides (289–293).

1. Consider a right angled triangle with hypotenuse D and sides K and B, as in Fig. 2.
2. Taking the sum of D and K, their difference, and multiplying yields $(D+K)(D-K) = D^2 - K^2 = B^2$. Note that $Z_K = D - K$ is the arrow.
3. This yields $\frac{Z_K(D+K)}{B} = B$, which allows to derive $D+K$ when given Z_K and B. From $D+K$ together with $Z_K = D-K$ we can derive D and K.
4. But here we are interested in the case where $\frac{B}{Z_K}$, rather than Z_K, is given, as this ratio will play the role of n. The previous formula can be rearranged as $B = \frac{D+K}{B/Z_K} = \frac{(B/Z_K)(D+K)}{(B/Z_K)^2} = \frac{(B/Z_K)(2K+Z_K)}{(B/Z_K)^2}$.
5. To reach the desired formula (the second of the three numbers above), we need to subtract 1 from the denominator. To retain the value of the fraction after the change of denominator, we need to remove the entire fraction, which equals B, from the numerator. We are left with $\frac{2(B/Z_K)K}{(B/Z_K)^2-1} = \frac{2nK}{n^2-1}$. This still equals the original $B = \frac{D+K}{B/Z_K}$, so given rational K and $n = \frac{B}{Z_K}$ we can obtain a rational B.
6. Now, multiplying B by $\frac{B}{Z_K}$ yields $D+K = \frac{2(B/Z_K)^2K}{(B/Z_K)^2-1}$. Subtracting K yields a rational $D = \frac{\left((B/Z_K)^2+1\right)K}{(B/Z_K)^2-1} = \frac{(n^2+1)K}{n^2-1}$.

The discussion goes on to the other formulas provided by the *Līlāvatī*:

1. Since $D^2 - K^2 = B^2$, we get $D+K = \frac{B^2}{D-K} = \frac{B^2}{Z_K}$.
2. Since half the sum of a sum and a difference is the larger term, we get $D = \frac{1}{2}((D+K)+(D-K)) = \frac{1}{2}\left(\frac{B^2}{Z_K} + Z_K\right)$. Similarly, $K = \frac{1}{2}\left(\frac{B^2}{Z_K} - Z_K\right)$.
3. If we add, rather than subtract, 1 to the denominator in step 5, we need to add the entire fraction (B) to the denominator. We get $\frac{2(B/Z_K)D}{(B/Z_K)^2+1} = B = \frac{D+K}{B/Z_K}$. From this follows that the triple D, $\frac{2nD}{n^2+1}$ and $\frac{(n^2-1)D}{n^2+1}$ forms a rational right angle triangle.
4. Alternatively, since $Z_K = D-K = \frac{2D}{n^2+1}$, we get $K = D - Z_K = D - \frac{2D}{n^2+1}$, and $B = \frac{B}{Z_K}Z_K = n\frac{2D}{n^2+1}$.

The following is the justification that for integer n and m, the three numbers $2nm$, $n^2 - m^2$ and $n^2 + m^2$ form a right angled triangle (293–297).

1. The proof is based on for Rsines and Rcosines of sums of angles: given two right angled triangles in a circle, the first having sides $(a, b, c) =$

$(D\sin\theta, D\cos\theta, D)$ and the second $(a', b', c') = \left(D\sin\varphi, D\cos\varphi, D\right)$, the following are the sides of a triangle whose angle is the sum of θ and φ, the given triangles' angles: $\left(ab' + ba'\right)/D$, $\left(aa' - bb'\right)/D$, $\left(aa' + bb'\right)/D$. This rule is attributed to Mādhava (see also Sarma et al. 2009).

2. Given a right angled triangle with integer sides (but not necessarily integer hypotenuse) n and m, applying this rule to this triangle and itself yields $2nm$, $n^2 - m^2$ and $n^2 + m^2$. Note, that in this application of the rule we do not divide by the hypotenuse, so given integer n and m, the new right angled triangle has all sides integers.
3. Another way of showing that the three numbers form a right angled triangle is to show that $(n^2 + m^2)^2 = (2nm)^2 + (n^2 - m^2)^2$.
4. First, $(n^2 + m^2)^2 = \left((n-m)^2 + 2nm\right)^2 = (n-m)^4 + (2nm)^2 + 4nm \times (n-m)^2$.
5. Second, $n^2 - m^2 = (n-m)(n+m)$, and therefore $(n^2 - m^2)^2 = (n-m)^2(n+m)^2$. Since $(n+m)^2 = (n-m)^2 + 4nm$, we get $(n^2 - m^2)^2 = (n-m)^4 + 4nm \times (n-m)^2$
6. Adding $(2nm)^2$ to $(n^2 - m^2)^2 = (n-m)^4 + 4nm \times (n-m)^2$, we obtain the right hand side of the last identity in step 4, and prove the claim of step 3.
7. Assuming $m = 1$, we get $2n$, $n^2 - 1$ and $n^2 + 1$. Applying the rule of three, we get variants of the three ways to produce a rational right angled triangle brought up in the *Līlāvatī*:
   - $K, \frac{(n^2-1)K}{2n}, \frac{(n^2+1)K}{2n}$
   - $\frac{2nB}{n^2-1}, B, \frac{(n^2+1)B}{n^2-1}$
   - $\frac{2nD}{n^2+1}, \frac{(n^2-1)D}{n^2+1}, D$

8. APPENDIX D

In the discussion of squaring numbers (Sarma 1975: 36), it is shown that the sum of odd numbers $1+3+\cdots+(2n-1)$ forms a square number by "wrapping" the rows of odd length around each other to form a square.

Later, in the discussion of taking square roots (Sarma 1975: 43 f.), the sum is proved again, this time from the formula of an arithmetic progression: the first and last terms together make $2n$, and the same goes for any pair with equal distance from the extremities. The number of terms is n, and so the sum is $2n \times \frac{n}{2} = n^2$. Then, the author states that he will show that the difference of squares is the product of the sum and difference of their roots. The argument is as follows:

1. It is first noted that in the sequence $1, 2, 3, \ldots$, the sum of each consecutive pair is smaller than the sum of the next consecutive pair by two.

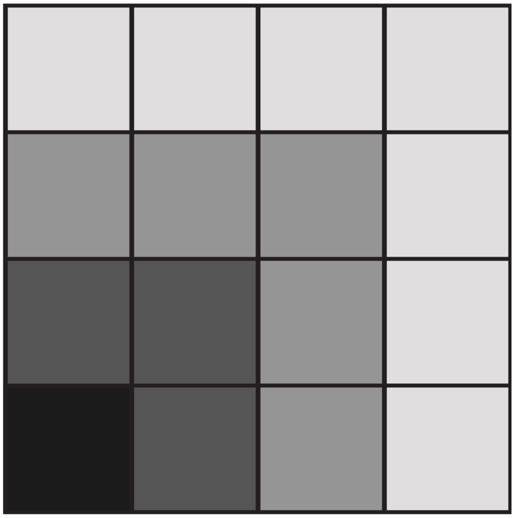

Figure 3: The sum of consecutive odd numbers is a square number.

2. These sums therefore produce the sequence $1, 3, 5, 7, ...$
3. Since the latter sequences sums to a square, each term of this sequence is a difference of consecutive squares ($3 = 2^2 - 1^2$, $5 = 3^2 - 2^2$, $\ldots$).
4. The verses stop at this point, but the rest is not hard to reconstruct: Taking the sum of some consecutive odd numbers, step 3 shows that they are the difference of squares (e.g., $3 + 5 + 7 = 4^2 - 1^2$).
5. The same sum is also the sum of pairs of consecutive numbers (in our example, $(1 + 2) + (2 + 3) + (3 + 4)$).
6. Using the same kind of reasoning as in summing arithmetic progressions, we can see that the latter is the product of a sum and a difference (in our example, $(4 + 1)(4 - 1)$). This can be done either arithmetically, by rewriting the sum as $n-k$ pairs that sum to $n+k$, or geometrically, by fitting together the shapes represented by $k+(k + 1)+ ... +(n - 1)$ and $(k + 1)+(k + 2)+ ... +n$ into a single rectangle that can be represented as $(n + k) + (n - 1 + k + 1) + ... + (k + 1 + n - 1) = (n + k)(n - k)$.

9. APPENDIX E

THIS IS A NON-EXPERT TRANSLATION OF THE VERSES comparing the rule of three to an analogical inference referred to as *anumāna* (Sarma 1975: 204–6; for more details see Hayashi 2000: section 3.2). The classical example of inference in this system is the following. One sees smoke on a mountain. One knows that smoke in the kitchen implies fire in the kitchen. So one concludes there's fire in the mountain. The mountain is one "side" (*pakṣa*), and the kitchen is the correlate "side" (*sapakṣa*). The mountain is subject to a certain quality or law (*dharma*), which applies also to the kitchen. The law consists of a given quality (*sadhana*, in this case smoke), and a required quality (*sādhya*, in this case fire).

This form of reasoning is applied to explain the rule of three. The terms here are *pramāṇa* (measure, marked here as *a*), *pramāṇaphala* (yield of measure, *b*), *icchā* (desired, *c*) and *icchāphala* (yield of desired, *d*). They satisfy the ratio $a : b :: c : d$.

I note with an asterisk (*) those places where my translation is most questionable due to grammatical difficulties in the original verses, but there are many other places where I am not certain of having captured the author's intentions correctly.

In the first verses, the terms are introduced, and the rule of three is equated with its calculation algorithm on the one hand, and with anumāna *inference on the other.*

1a–1d: When a *phala* [*d*] to be known is some quantity multiplied by something and divided by something, it's nothing but the rule of three.

2a–2b: And that, on the other hand, does not exceed the method of *anumāna*.

2c–3b: Because a *phala* [*d*] got by means of calculation with the three quantities known as *phala* [*b*], *pramāṇa* [*a*] and *icchā* [*c*] has its origin in the rule of three.

Next, anumāna *terminology is being introduced. The* icchā (c) *and the unknown result* (d) *are identified as the "side"* (pakṣa)*, characterized by its distinct (not yet known)* dharma.

3c–3d: The *phala* to be known [*d*] is considered here as the required [*sādhya*] *dharma*.

4a–4b: The *icchā* quantity [*c*], which is one side [*pakṣa*], is endowed with the *phala* to be known [*d*].

4c–4d: Indeed, the characteristic feature of the required *dharma* is indicated by being on the side that has the dharma [*dharmipakṣatā*].

5a–5b: Here, the reason for the pervasion of the given [*sādhana*] by the required is clear:

5c–5d: It has the dharma of the side.* The unknown [*d*] should indeed be placed with the *icchā* quantity [*c*].

The correlate "side," of the pramāṇa (a) *and its* phala (b)*, is pervaded by the same* dharma *as the previous side. This is because the* pramāṇa (a) *and* icchā (c) *are of the same kind, but on different sides.*

6.a–6.b: But it is clear that the same kind [*jātīya*] is in the *pramāṇa* [*a*] as in the *icchā* quantity [*c*].

6.c–6.d: However, the given is always in the *pramāṇa* [*a*], not in the required.

7.a–7.d: So the correlate side [*sapakṣa*], as the side of the unsettled *phala* [*d*] and *icchā* [*c*], is indeed endowed with a consequence [*anubhava*] of a rule [*niyama*] due to the pervasion of the given by the required.

8.a–8.d: While the *icchā* quantity [*c*] is endowed with the *phala* to be known [*d*], the *dharma* pervaded by the *phala* [*b*] and the *phala* [*b*] have the *pramāṇa* [*a*].*

The same relation is also explained as the result of repeated observation (induction), rather than by means of the essences involved.

9.a–9.b: Moreover, because observation is intermixed with reasoning [*tarka*], the pervasion is to be ascertained.

9.c–9.d: "Where there's so much *pramāṇa* [*a*], there there's that much of its *phala* [*b*],"

10.a–10.b: it is said. So their pervasion is ascertained from repeated observation.

Given a certain dharma *relating* a *and* b, *and given some* c *of the same kind as* a, *we can derive* d *according to that* dharma. *If* c *is not of the same kind (according to my reading: not measured by the same units), then* c *has to be converted to the same kind as* a.

10.c–10.d: Wherever the likeness [*sarūpya, c*] of the *pramāṇa* [*a*] is seen, in whatever object,

11.a–11.d: even if the *phala* [*d*] is not manifest there, it is considered like the *pramāṇa* [*a*], and the likeness [*c*] is considered according to their appropriate *dharma*.

12.a–12.b: Moreover, if there's no *phala* [*d*] because of lack of likeness,

12.c–12.d: their likeness and their having the same kind [*sajātīyatva*] is desired,

13.a–13.b: as wherever some *dharma* pervaded by the *phala* is inferred,

13.c–13.d: if this very [likeness] is seen elsewhere, there the *phala* [*d*] would be derived.

Finally, the relations are restated, and the common dharma *is expressed in terms of equality of proportions in the rule of three.*

14.a–14.b: The *pramāṇa* [*a*] of the *dharma* pervaded by the *phala* [*b*] and of its *phala* [*b*]

14.c–14.d: is the place where the pervasion is perceived. The *icchā* [*c*] is endowed with the desired *phala* [*d*].

15.a–15.b: Where the *phala* [*d*] is to be approached, there [it is] in the *icchā* [*c*], like in the *pramāṇa* [*a*].*

15.c–15.d: In the *icchā* quantity [*c*], which is the side [*pakṣa*], would be its unsettled *phala* [*d*].

16.a–16.b: In the *pramāṇa* quantity [*a*] the given is always understood by the required,

16.c–16.d: as the *phala* [*b*] is a part determined by something.

17.a–17.b: Having obtained that the *pramāṇa* [*a*] is pervaded by the *phala* [*b*], again by that knowledge,

17.c–17.d: the *phala* [*d*] in the *icchā* quantity [*c*] is obtained thus by the calculators:

18.a–18.b: The known *phala* [b] being such part or rather such multiple of the *pramāṇa* [*a*],

18.c–18.d: in the *icchā* quantity [*c*], the *phala* [*d*] is considered as that part or rather that multiple.

10. APPENDIX F

PROOF OF HERON'S FORMULA for the area of a triangle (Sarma 1975: 321–5).

1. Statement: Given a triangle with sides a, c and base b, the area is the root of the product of $\frac{a+b+c}{2} - a$, $\frac{a+b+c}{2} - c$, $\frac{a+c}{2} - \frac{b}{2}$ and $\frac{a+c}{2} + \frac{b}{2}$.
2. The first two terms equal $\frac{b}{2} - \frac{a-c}{2}$ and $\frac{b}{2} + \frac{a-c}{2}$. Their product is $\left(\frac{b}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2$.

   The product of the last two terms is $\left(\frac{a+c}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.
3. Let l (*lamba*) be the height on the base, and let b_1, b_2 be the parts of the base on both sides of the height. From the Pythagorean theorem we have $l^2 = a^2 - {b_1}^2 = c^2 - {b_2}^2 = \frac{a^2+c^2}{2} - \frac{{b_1}^2+{b_2}^2}{2}$. Moreover, $a^2 - c^2 = {b_1}^2 - {b_2}^2$.

4. By quadratic identities, this implies $l^2 = \left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2 - \left(\frac{b_1-b_2}{2}\right)^2$. By rearrangement we get $\left(\frac{b_1+b_2}{2}\right)^2 + \left(\frac{b_1-b_2}{2}\right)^2 = \left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2 - l^2$ and $\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2 = \left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2 - l^2$.

5. From the identity in step 3 and a quadratic identity, $\left(\frac{a+c}{2}\right)^2 \times \frac{\left(\frac{a-c}{2}\right)^2}{\left(\frac{b_1+b_2}{2}\right)^2} = \frac{\left(\frac{a^2-c^2}{4}\right)^2}{\left(\frac{b_1+b_2}{2}\right)^2} = \frac{\left(\frac{{b_1}^2-{b_2}^2}{4}\right)^2}{\left(\frac{b_1+b_2}{2}\right)^2} = \left(\frac{b_1-b_2}{2}\right)^2$.

6. Since multiplying and dividing by the same number preserves the result, $\left(\frac{b_1+b_2}{2}\right)^2 \times \frac{\left(\frac{a-c}{2}\right)^2}{\left(\frac{b_1+b_2}{2}\right)^2} = \left(\frac{a-c}{2}\right)^2$.

7. Taking the difference of steps 5 and 6, we get $\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) \times \frac{\left(\frac{a-c}{2}\right)^2}{\left(\frac{b_1+b_2}{2}\right)^2} = \left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2$.

8. The terms are thus related to each other by the rule of three:
   - $\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) : \left(\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right) :: \left(\frac{b_1+b_2}{2}\right)^2 : \left(\frac{a-c}{2}\right)^2$.
   - $\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) : \left(\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right) :: \left(\frac{a+c}{2}\right)^2 : \left(\frac{b_1-b_2}{2}\right)^2$.
   - In other words, the left hand ratio turns each sum to the other difference.

9. - From step 4, $\frac{\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) - \left(\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right)}{\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2} = \frac{l^2}{\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2}$.
   - From step 7, this equals $\frac{\left(\frac{b_1+b_2}{2}\right)^2}{\left(\frac{a-c}{2}\right)^2} - 1 = \frac{\left(\frac{a+c}{2}\right)^2}{\left(\frac{b_1-b_2}{2}\right)^2} - 1$.

10. Here this thread of the argument is interrupted (my best reading suggests that the author changed strategy due to computational errors), and the discussion continues by analogy to the following situation: Consider two intersecting circles. The chord C of the intersection is bisected by the line connecting the centers of the circles. This line contains the two radii R_1 and R_2, which, in turn, contain the arrows Z_1 and Z_2. If $R_1 > R_2$, then

$Z_2 > Z_1$.

11. We also have $(R_1 - Z_1)Z_1 = \left(\frac{C}{2}\right)^2 = (R_2 - Z_2)Z_2$. Therefore $(R_1 - Z_1) : (R_2 - Z_2) :: Z_2 : Z_1$. Therefore, $(R_1 - Z_1) : (R_2 - Z_2) :: (R_1 - Z_1 - Z_2) : (R_2 - Z_2 - Z_1)$.
12. As a result we have $(R_1 - Z_1 - Z_2) \times \frac{Z_1+Z_2}{(R_1-Z_1-Z_2)+(R_2-Z_2-Z_1)} = Z_2$ (all this is referred to the *Aryabhaṭīya*).
13. This is the end of the verses, but the following analogy may be reconstructed: Compare the first proportion of step 8 to the first proportion of step 11.
    - $\left(\frac{a+c}{2}\right)^2$ plays the role of R_1.
    - $\left(\frac{b_1-b_2}{2}\right)^2$ plays the role of R_2.
    - $\left(\frac{b_1+b_2}{2}\right)^2$ plays the role of Z_1 on the left hand side and that of Z_2 on the right.
    - $\left(\frac{a-c}{2}\right)^2$ plays the role of Z_2 on the left hand side and that of Z_1 on the right.
14. By analogy to $(R_1 - Z_1) \times \frac{Z_2-Z_1}{(R_1-Z_1)-(R_2-Z_2)} = Z_2$ (which can be obtained from step 11 by a reasoning similar to the one leading to step 12), one obtains:

$$\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) \times \frac{\left(\frac{b_1+b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2}{\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) - \left(\left(\frac{b_1-b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right)} = \left(\frac{b_1+b_2}{2}\right)^2.$$

15. By step 4, the denominator in the previous step is l^2, so we get

$$\left(\left(\frac{a+c}{2}\right)^2 - \left(\frac{b_1+b_2}{2}\right)^2\right) \times \left(\left(\frac{b_1+b_2}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right) = \left(\frac{b_1+b_2}{2}\right)^2 \times l^2.$$

The left hand side is the squared formula from steps 1–2, and the right hand side is the square of the height times the square of half the base, namely the square of the area of the triangle.

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